



# ON SURFACES OF CLASS VII WITH NUMERICALLY ANTICANONICAL DIVISOR

Georges Dloussky

► To cite this version:

Georges Dloussky. ON SURFACES OF CLASS VII WITH NUMERICALLY ANTICANONICAL DIVISOR. American Journal of Mathematics, 2006. hal-01255762

**HAL Id: hal-01255762**

**<https://hal.science/hal-01255762>**

Submitted on 14 Jan 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On surfaces of class $VII_0^+$ with numerically anticanonical divisor

Georges Dloussky

## Abstract

We consider minimal compact complex surfaces  $S$  with Betti numbers  $b_1 = 1$  and  $n = b_2 > 0$ . A theorem of Donaldson gives  $n$  exceptional line bundles. We prove that if in a deformation, these line bundles have sections,  $S$  is a degeneration of blown-up Hopf surfaces. Besides, if there exists an integer  $m \geq 1$  and a flat line bundle  $F$  such that  $H^0(S, -mK \otimes F) \neq 0$ , then  $S$  contains a Global Spherical Shell. We apply this last result to complete classification of bihermitian surfaces.

## Contents

<b>0</b>	<b>Introduction</b>	<b>1</b>
<b>1</b>	<b>Exceptional line bundles and degeneration of blown-up Hopf surfaces</b>	<b>4</b>
1.1	Exceptional line bundles . . . . .	4
1.2	Families of exceptional or flat line bundles . . . . .	6
1.3	Degeneration of blown-up Hopf surfaces . . . . .	8
<b>2</b>	<b>Surfaces with a cycle of rational curves</b>	<b>14</b>
2.1	Surfaces with numerically m-anticanonical divisor . . . . .	14
2.2	The reduction lemma . . . . .	15
2.3	Characterization of Inoue-Hirzebruch surfaces . . . . .	17
2.4	Surfaces with singular rational curve . . . . .	19
2.5	Cycles with at least two rational curves . . . . .	21
<b>3</b>	<b>On classification of bihermitian surfaces</b>	<b>25</b>
3.1	Conformal and complex structures . . . . .	25
3.2	Numerically anticanonical divisor of a bihermitian surface . . . .	27
3.3	Bihermitian surfaces with odd first Betti number . . . . .	28

## 0 Introduction

A minimal compact complex surface  $S$  is said to be of the class  $VII_0$  of Kodaira if the first Betti number satisfies  $b_1(S) = 1$ . A surface  $S$  is of class  $VII_0^+$  if moreover  $n := b_2(S) > 0$ ; these surfaces admit no nonconstant meromorphic functions. The major problem in classification of non-kählerian surfaces is to achieve the classification of surfaces  $S$  of class  $VII_0^+$ . All known surfaces of this

class contain Global Spherical Shells (GSS), i.e. admit a biholomorphic map  $\varphi : U \rightarrow V$  from a neighbourhood  $U \subset \mathbb{C}^2 \setminus \{0\}$  of the sphere  $S^3 = \partial B^2$  onto an open set  $V$  such that  $\Sigma = \varphi(S^3)$  does not disconnect  $S$ . Are there other surfaces ? In first section we investigate the general situation: A theorem of Donaldson [13] gives a  $\mathbb{Z}$ -base  $(E_i)$  of  $H^2(S, \mathbb{Z})$ , such that  $E_i E_j = -\delta_{ij}$ . These cohomology classes can be represented by line bundles  $L_i$  such that  $K_S L_i = L_i^2 = -1$ . Indeed, these line bundles generalize exceptional curves of the first kind, and since  $S$  is minimal, they have no section. Over the versal deformation  $\mathcal{S} \rightarrow B$  of  $S$  these line bundles form families  $\mathcal{L}_i$ . We propose the following conjecture which can be easily checked for surfaces with GSS:

**Conjecture 1:** Let  $S$  be a surface in class  $VII_0^+$  and  $\mathcal{S} \rightarrow B$  be the versal deformation of  $S \simeq S_0$  over the ball of dimension  $h^1(S, \Theta)$ . Then there exists  $u \in \Delta$ ,  $u \neq 0$ , and flat line bundles  $F_i$  such that  $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$  for  $i = 0, \dots, n-1$ .

The main result of section 1 is (see theorem (13)),

**Theorem 0. 1** *Let  $S$  be a surface in class  $VII_0^+$  and  $\mathcal{S} \rightarrow B$  its versal deformation. If there exists  $u \in B$  and flat line bundles  $F_i \in H^1(S, \mathbb{C}^*)$  such that  $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$  for  $i = 0, \dots, n-1$ , then there is a non empty Zariski open set  $U \subset B$  such that for all  $u \in U$ ,  $S_u$  is a blown-up Hopf surface. In particular,  $S$  is a degeneration of blown-up Hopf surfaces.*

If a surface is a degeneration of blown-up Hopf surfaces, the fundamental group of a fiber is isomorphic to  $\mathbb{Z} \times \mathbb{Z}_l$ , hence taking a finite covering, once obtains a surface obtained by degeneration of blown-up primary Hopf surfaces. Notice that a finite quotient of a surface of class  $VII_0^+$  containing a GSS still contains a GSS [8].

**Conjecture 2:** Let  $S$  be a surface of class  $VII_0^+$ . If  $S$  is a degeneration of blown-up primary Hopf surfaces, then  $S$  contains a cycle of rational curves.

A surface admitting a numerically anticanonical (NAC) divisor (see (20)), contains a cycle of rational curves. In section 2, we shall prove

**Theorem 0. 2** *Let  $S$  be a surface of class  $VII_0^+$ . If  $S$  admits a NAC divisor, then  $S$  contains a GSS.*

It is a weak version of

**Conjecture 3 (Nakamura [21]).** Let  $S$  be a surface of class  $VII_0^+$ . If  $S$  contains a cycle of rational curves,  $S$  contains a GSS.

The proof is based on the fact that in  $H^2(S, \mathbb{Z})$ , a curve is equivalent to a class of the form  $L_i - \sum_{j \in I} L_j$ , with  $I \neq \emptyset$ . Intuitively  $L_i$  represents an exceptional curve of the first kind and  $C$  is then equivalent to an exceptional curve of the first kind blown-up several times ( $\text{Card}(I)$  times). It explains why curves have self-intersection  $\leq -2$ . We recover a characterization of Inoue-Hirzebruch surfaces by Oeljeklaus, Toma & Zaffran [19]:

**Theorem 0. 3** *Let  $S$  be a surface of class  $VII_0$  with  $b_2(S) > 0$ . Then  $S$  is a Inoue-Hirzebruch surface if and only if there exists two flat line bundles  $F_1$ ,*

$F_2$ , two twisted vector fields  $\theta_1 \in H^0(S, \Theta \otimes F_1)$ ,  $\theta_2 \in H^0(S, \Theta \otimes F_2)$ , such that  $\theta_1 \wedge \theta_2(p) \neq 0$  at at least one point  $p \in S$ .

In section 3 we apply results of section 2 to complete the classification of bihermitian 4-manifolds  $M$  (see [1],[2] [22]), when  $b_1(M) = 1$  and  $b_2(M) > 0$ : A bihermitian surface is a riemannian oriented connected 4-manifold  $(M, g)$  endowed with two integrable almost complex structures  $J_1, J_2$  inducing the same orientation, orthogonal with respect to  $g$  and independent i.e.  $J_1(x) \neq \pm J_2(x)$  for at least one point  $x \in M$ . This structure depends only on the conformal class  $c$  of  $g$ . A bihermitian surface is strongly bihermitian if  $J_1(x) \neq \pm J_2(x)$  for every point  $x \in M$ . The key observation is that under these assumptions,  $(M, J_i)$ ,  $i = 1, 2$  admit a numerically anticanonical divisor.

**Theorem 0. 4** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface with odd first Betti number.*

1) *If  $(M, c, J_1, J_2)$  is strongly bihermitian (i.e  $\mathcal{D} = \emptyset$ ), then the complex surfaces  $(M, J_i)$  are minimal and either a Hopf surface covered by a primary one associated to a contraction  $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  of the form*

$$F(z_1, z_2) = (\alpha z_1 + s z_2^m, a \alpha^{-1} z_2),$$

$$\text{with } a, s \in \mathbb{C}, 0 < |\alpha|^2 \leq a < |\alpha| < 1, (a^m - \alpha^{m+1})s = 0,$$

*or else  $(M, J_i)$  are Inoue surfaces  $S_{N,p,q,r;t}^+, S_{N,p,q,r}^-$ .*

2) *If  $(M, c, J_1, J_2)$  is not strongly bihermitian, then  $\mathcal{D}$  has at most two connected components,  $(M, J_i)$ ,  $i = 1, 2$ , contain GSS and the minimal models  $S_i$  of  $(M, J_i)$  are*

- *Surfaces with GSS of intermediate type if  $\mathcal{D}$  has one connected component*
- *Hopf surfaces of special type (see [22] 2.2), Inoue (parabolic) surfaces or Inoue-Hirzebruch surfaces if  $\mathcal{D}$  has two connected components.*

*Moreover, the blown-up points belong to the NAC divisors.*

If moreover the metric  $g$  is anti-self-dual (ASD), we obtain

**Corollary 0. 5** *Let  $(M, c, J_1, J_2)$  be a compact ASD bihermitian surface with odd first Betti number. Then the minimal models of the complex surfaces  $(M, J_i)$ ,  $i = 1, 2$ , are*

- *Hopf surfaces of special type (see [22] 2.2),*
- *(parabolic) Inoue surfaces or*
- *even Inoue-Hirzebruch surfaces.*

*Moreover, the blown-up points belong to the NAC divisors.*

**Remark 0. 6** Throughout the paper we shall use the following terminology:

1) A surface for which exists a nontrivial divisor  $D$  such that  $D^2 = 0$  will be called a **Enoki surface**, they are obtained by holomorphic compactification of

an affine line bundle over an elliptic curve by a cycle  $D$  of rational curves [14]; otherwise they are associated to contracting holomorphic germs of maps

$$F(z_1, z_2) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, t z_2)$$

[6] [9]. A Enoki surface with an elliptic curve will be called briefly a **Inoue surface** (= **parabolic Inoue surface**): they are obtained by holomorphic compactification of a line bundle over an elliptic curve by a cycle of rational curves  $D$ ; otherwise they are associated to the contracting germs of maps

$$F(z_1, z_2) = (t^n z_1 z_2^n, t z_2).$$

For all these surfaces, the sum of opposite self-intersections of the  $n = b_2(S)$  rational curves  $D_0, \dots, D_{n-1}$  is  $\sigma_n(S) := -\sum_{i=0}^{n-1} D_i^2 = 2n$ .

2) A surface with  $2n < \sigma_n(S) < 3n$  will be called an **intermediate surface**;  
3) A surface with  $\sigma_n(S) = 3n$  is called a **Inoue-Hirzebruch (IH) surface** ([15] or [7] for a construction by contracting germs of mappings). An **even** (=hyperbolic) (resp. **odd** (=half)) **IH surfaces** has two (resp. one) cycle of rational curves.

**We shall assume throughout the article that  $S$  admits no nonconstant meromorphic functions.**

**Acknowledgments:** I am grateful for helpful discussions with Akira Fujiki, Massimiliano Pontecorvo, and Vestislav Apostolov during the preparation of the third part of this article. There is a gap in the proof of Corollary 2, p 425 in [2], theorems 4.1 and 5.2 in [22]. The statements are too restrictive and omit Inoue-Hirzebruch surfaces.

## 1 Exceptional line bundles and degeneration of blown-up Hopf surfaces

Let  $S$  be a surface in class  $\text{VII}_0^+$  with  $n = b_2(S)$ . Since  $S$  is not algebraic,  $A^2 \leq 0$  for every divisor  $A$ . By adjunction formula it is easy to deduce that for every irreducible curve  $C$ ,  $KC \geq 0$ ,  $C^2 \leq -2$  if  $C$  is a regular rational curve and  $C^2 \leq 0$  if  $C$  is a rational curve with a double point or an elliptic curve. It is well known that  $S$  contains at most  $n$  rational curves, and at most one elliptic curve. By Hirzebruch index theorem,  $b^- = b_2(S)$  whence the intersection form  $Q : H^2(S, \mathbb{Z})/\text{Torsion} \rightarrow \mathbb{Z}$  is negative definite.

### 1.1 Exceptional line bundles

An irreducible curve which satisfies the two conditions  $KC = C^2 = -1$  is an exceptional curve of the first kind; we generalize the notion to line bundles.

**Definition 1.7** 1) A line bundle  $L \in H^1(S, \mathcal{O}^*)$  is called an **exceptional line bundle** (of the first kind) if  $KL = L^2 = -1$ .

2) An effective divisor  $E$  is called an exceptional divisor (of the first) kind if  $E$  is the inverse image  $\Pi^*C$  of an exceptional curve of the first kind  $C$  by a finite number of blowing-ups  $\Pi$ . Equivalently it is an effective reduced divisor which may be blown-down onto a regular point.

Using the fact that for a blowing-up  $\Pi : S \rightarrow S'$ ,  $K_S = \Pi^*K_{S'} + C$  and the projection formula ([5] p11), it is easy to check that if  $E$  is an exceptional divisor, then  $[E]$  is an exceptional line bundle. Moreover, the inverse image  $\Pi^*L$  of an exceptional line bundle  $L$  by a finite sequence of blowing-ups is still an exceptional line bundle.

The following theorem has been proved by I. Nakamura when  $S$  contains a cycle of rational curves [21] (1.7). It should be noticed that any surface with  $b_1(S) = 1$  and  $b_2(S) = 0$  is minimal and  $H^1(S, \mathbb{C}^*) \simeq H^1(S, \mathcal{O}^*)$  [17] II, p699.

**Theorem 1. 8** *Let  $S$  be a (not necessarily minimal) compact complex surface such that  $b_1(S) = 1$ , with second Betti number  $n = b_2(S) > 0$ . Then there exists  $n$  exceptional line bundles  $L_j$ ,  $j = 0, \dots, n-1$ , unique up to torsion by a flat line bundle  $F \in H^1(S, \mathbb{C}^*)$  such that:*

- $E_j = c_1(L_j)$ ,  $0 \leq j \leq n-1$  is a  $\mathbb{Z}$ -basis of  $H^2(S, \mathbb{Z})$ ,
- $K_S L_j = -1$  and  $L_j L_k = -\delta_{jk}$ ,
- $K_S = L_0 + \dots + L_{n-1}$  in  $H^2(S, \mathbb{Z})$
- For every  $i = 0, \dots, n-1$  and for every flat line bundle  $F \in H^1(S, \mathbb{C}^*)$ ,

$$\mathcal{X}(L_i \otimes F) = 0.$$

- If  $h^0(S, L_i \otimes F) \neq 0$ , there exists an exceptional divisor  $C_i$  and a (perhaps trivial) flat effective divisor  $P_i$  such that  $L_i \otimes F = [C_i + P_i]$ .

Proof: 1) By Donaldson theorem [13], there exists a  $\mathbb{Z}$ -basis  $(E_i)_i$ ,  $0 \leq i \leq n-1$ , of  $H^2(S, \mathbb{Z})/\text{Torsion}$  such that  $E_i E_j = -\delta_{i,j}$ . Moreover, since  $p_g = h^2(S, \mathcal{O}_S) = 0$ , the exponential exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$  yields line bundles  $L_i$  such that  $E_i = c_1(L_i)$ . The line bundles  $L_i$  are unique up to tensor product by flat line bundles. For a surface of class VII, the group of flat line bundles is  $H^1(S, \mathbb{C}^*)$ . Let  $c = \sum n_i E_i \in H^2(S, \mathbb{Z})$ . Then  $c^2 = -\sum n_i^2$ , therefore  $c^2 = -1$  if and only if  $c = \pm E_i$ . Replacing if necessary  $L_i$  by  $L_i^{-1}$  we may suppose that  $KL_i \leq 0$  for  $i = 0, \dots, n-1$ . By Riemann-Roch formula

$$(*) \quad \mathcal{X}(L_i \otimes F) = \mathcal{X}(\mathcal{O}_S) + \frac{1}{2}(L_i^2 - KL_i) = \frac{1}{2}(-1 - KL_i) \in \mathbb{Z},$$

therefore  $KL_i \leq -1$ . Since  $(L_i)$  is a  $\mathbb{Z}$ -base of  $H^2(S, \mathbb{Z})$ ,  $K = \sum_i k_i L_i$  with  $k_i = -KL_i \geq 1$ . From  $-n = K^2 = -\sum_{i=0}^{n-1} k_i^2$  we deduce that  $k_i = 1$  for  $i = 0, \dots, n-1$ . From (\*) we obtain

$$\mathcal{X}(L_i \otimes F) = 0.$$

2) If  $h^0(S, L_i \otimes F) > 0$ , then  $L_i \otimes F = [C_i]$  where  $C_i$  is an effective divisor. Let

$$C_i = n_1 G_1 + \cdots + n_p G_p$$

be a decomposition into irreducible components.

- If  $G$  is an elliptic curve or a rational curve with a double point then  $KG = -G^2 \geq 0$ .
- If  $G$  is a nonsingular rational curve,  $KG = -2 - G^2 \geq -1$  and  $KG = -1$  if and only if  $G$  is an exceptional curve of the first kind.

Therefore the condition

$$-1 = KC_i = \sum_i n_i KG_i$$

implies that there is an exceptional curve of the first kind, say  $G_p$ . Now we prove the result by induction on  $n = b_2(S) \geq 1$ .

If  $n = 1$ , there is only one exceptional line bundle  $L$  and if  $h^0(S, L \otimes F) \neq 0$ ,  $S$  is not minimal, hence a blow-up of a surface  $S'$  with  $b_2(S') = 0$ . Then  $L \otimes F = [C + P]$  where  $C$  is an exceptional curve of the first kind and  $P = 0$  (if  $S'$  has no curve) or  $P$  is flat perhaps not trivial (if  $S'$  is a Hopf surface). Suppose that  $n > 1$ :

- If  $G_p \sim E_i$ , then  $-F = [n_1 G_1 + \cdots + n_{p-1} G_{p-1} + (n_p - 1)G_p]$  is a flat line bundle and  $L'_i = L_i \otimes F = [G_p]$ .
- If  $G_p \sim E_j$ ,  $j \neq i$ , there is a flat line bundle  $F$  such that  $L'_j = L_j \otimes F = [G_p]$ .

Therefore, replacing  $L_j$  by  $L'_j$ , and changing if necessary the numbering we may suppose that  $L_{n-1} = [C_{n-1}]$  with  $C_{n-1}$  an exceptional curve of the first kind. Let  $\Pi : S \rightarrow S'$  be the blowing-down of  $C_{n-1}$ . Since  $L_i$ ,  $i \neq n-1$  is trivial in a neighbourhood of  $C_{n-1}$ ,  $L'_i = \Pi_* L_i$  is a line bundle such that  $L_i = \Pi^* \Pi_* L_i$  and we check easily, by projection formula, that  $(L'_i)_{0 \leq i \leq n-2}$  is a family of exceptional line bundles. If  $h^0(L_i) \neq 0$ , then  $h^0(L'_i) \neq 0$ , whence by induction hypothesis,  $L'_i = [C'_i + P'_i]$  with  $C'_i$  an exceptional divisor and  $P'_i$  an effective flat divisor. Therefore  $L_i = \Pi^* L'_i = [\Pi^* C'_i + \Pi^* P'_i]$ .  $\square$

## 1.2 Families of exceptional or flat line bundles

If  $S$  is minimal then, for any  $i = 0, \dots, n-1$  and for any flat line bundle  $F$ ,  $h^0(S, L_i \otimes F) = 0$ . In all known examples  $S$  has a deformation into a non minimal one, hence we consider now versal deformation of  $S$ .

Let  $\Pi : \mathcal{S} \rightarrow B$  be the versal deformation of  $S \simeq S_0$ , where  $B$  is the unit ball of  $\mathbb{C}^N$ ,  $N = \dim H^1(S, \Theta)$ . Standard arguments of spectral sequences yield

**Proposition 1. 9** For  $i = 0, \dots, n-1$ , there exist line bundles  $\pi_i : \mathcal{L}_i \rightarrow \mathcal{S}$  such that for every  $u \in B$ , the restriction  $\pi_{i,u} : L_{i,u} \rightarrow S_u$  is an exceptional line bundle,

Moreover, if  $\mathcal{K} \rightarrow \mathcal{S}$  is the relative canonical bundle, we have

$$\mathcal{K} \sim \bigotimes_{i=0}^{n-1} \mathcal{L}_i$$

in  $H^2(\mathcal{S}, \mathbb{Z}) \simeq H^2(S_0, \mathbb{Z})$ .

Since  $\mathbb{C}^*$  is commutative any representation  $\rho : \pi_1(S) \rightarrow GL(1, \mathbb{C}) \simeq \mathbb{C}^*$  factorize through  $H_1(S, \mathbb{Z})$ , therefore any representation (hence any flat line bundle), is defined by  $\gamma \mapsto f$  with  $f \in \mathbb{C}^*$ . We shall denote by  $F^f \in H^1(S, \mathbb{C}^*) \simeq \mathbb{C}^*$  this line bundle and we have defined a group morphism

$$\begin{array}{ccc} \varphi : & \mathbb{C}^* & \rightarrow H^1(S, \mathbb{C}^*) \\ & \lambda & \mapsto F^\lambda \end{array}$$

**Lemma 1. 10** 1) For any holomorphic function  $f : B \rightarrow \mathbb{C}^*$  there exists a unique flat line bundle  $\mathcal{F}^f \rightarrow \mathcal{S}$  such that  $(\mathcal{F}^f)_{|S_u} = F^{f(u)}$ .

2) There exists over  $\mathcal{S} \times \mathbb{C}^*$  a flat line bundle, called the **tautological flat line bundle**  $\mathcal{F}$  such that for any  $(u, \lambda) \in B \times \mathbb{C}^*$ ,  $\mathcal{F}_{S_u \times \{\lambda\}} = F^\lambda$ .

Proof: Let  $\omega : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$  be the family of universal covering spaces of  $\mathcal{S}$ . Then the fundamental group  $\pi_1(S)$  operates diagonally on  $\tilde{\mathcal{S}} \times \mathbb{C}$  by  $\gamma.(p, z) = (\gamma.p, f(\Pi\omega(p))z)$ . The quotient manifold is  $\mathcal{F}^f$ .

A similar construction gives the tautological flat line bundle  $\mathcal{F}$ .  $\square$

**Examples 1. 11** 1) Suppose that  $S = S(F)$  is a primary Hopf surface defined by

$$F(z) = (\alpha_1 z_1 + s z_2^m, \alpha_2 z_2), \quad 0 < |\alpha_1| \leq |\alpha_2| < 1, \quad (\alpha_2^m - \alpha_1)s = 0.$$

If  $s = 0$  (resp.  $s \neq 0$ ),  $S$  contains at least two elliptic curves  $E_1, E_2$  (resp. only one elliptic curve  $E_2$ ), where

$$E_1 = \{z \neq 0 \mid z_1 = 0\} / \{\alpha_2^p \mid p \in \mathbb{Z}\}, \quad E_2 = \{z \neq 0 \mid z_2 = 0\} / \{\alpha_1^p \mid p \in \mathbb{Z}\}.$$

Then for  $i = 1, 2$ ,  $\varphi(\alpha_i) = [E_i]$ . In fact, if  $S$  is a diagonal Hopf surface, the cocycle of the line bundle associated to  $[E_i]$  is given by

$$(z_1, z_2, \lambda) \sim (\alpha_1 z_1, \alpha_2 z_2, \alpha_i \lambda).$$

2) Following [6], let  $S = S(F)$  be the minimal surface containing a GSS with  $b_2(S) > 0$  defined by

$$F(z) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, t z_2).$$



Then  $S$  contains a cycle of rational curves  $\Gamma = D_0 + \cdots + D_{n-1}$  such that  $D_i^2 = -2$ ,  $i = 0, \dots, n-1$  and  $\Gamma^2 = 0$ . Let  $t = \text{tr} DF(0) \neq 0$  be the trace of the surface, then  $\varphi(t) = [\Gamma]$ . In fact the equation of  $\Gamma$  is  $z_2 = 0$ .

If in the expression of  $F$ , there is at least one index  $i$  such that  $a_i \neq 0$ ,  $S$  has no elliptic curve. If  $M(S)$  is the intersection matrix of the rational curves then  $\det M(S) = 0$ , hence the curves do not generate  $H^2(S, \mathbb{Z})$ ; for every  $m \geq 1$ , every  $F$  flat,  $H^0(S, -mK + F) = 0$ , therefore there is no NAC divisor.

If  $F(z) = (t^n z_1 z_2^n, t z_2)$ , i.e.  $S$  is a Inoue (parabolic) surface,  $S$  contains an elliptic curve  $E$  and  $-K = [E + \Gamma]$ .

3) Following [15], let  $S = S_M = \mathbb{H} \times \mathbb{C}/G_M$  with  $M \in SL(3, \mathbb{Z})$  a unimodular matrix with eigenvalues  $\alpha, \beta, \bar{\beta}$  such that  $\alpha > 1$ ,  $\beta \neq \bar{\beta}$ . Denote by  $(a_1, a_2, a_3)$  a real eigenvector associated to  $\alpha$  and  $(b_1, b_2, b_3)$  an eigenvector associated to  $\beta$ . It can be easily checked that  $(a_1, b_1)$ ,  $(a_2, b_2)$  and  $(a_3, b_3)$  are linearly independent over  $\mathbb{R}$ . Let  $G_M$  generated by

$$g_0 : (w, z) \mapsto (\alpha w, \beta z),$$

$$g_i : (w, z) \mapsto (w + a_i, z + b_i) \quad \text{for } i = 1, 2, 3.$$

If  $G$  is generated by  $g_i$ ,  $i = 1, 2, 3$ ,  $\omega = dw \wedge dz$  is invariant under  $G$  hence yields a 2-form on  $\mathbb{H} \times \mathbb{C}/G$ . Moreover,  $g_0^* \omega = \alpha \beta \omega$ , hence yields a non-vanishing twisted 2-form over  $S_M$  and  $K = F^{1/\alpha\beta}$ . A line bundle has no section for there is no curve.

### 1.3 Degeneration of blown-up Hopf surfaces

All surfaces containing GSS are degeneration of blown-up primary Hopf surfaces as it can be easily checked using contracting germs of mappings. We show in this section that if a surface can be deformed into a non minimal one then over a Zariski open set in the base of the versal deformation, there are blown-up Hopf surfaces.

We need a lemma comparing the versal deformation of a surface  $S$  with the versal deformation of a blowing-up  $S'$  of  $S$ .

**Lemma 1. 12** *Let  $S$  be a compact complex surface of the VII-class (not necessarily minimal), let  $\Pi : S' \rightarrow S$  be the blowing-up of  $S$  at the point  $z_0$  and*

$$e_{z_0} : H^0(S, \Theta) \rightarrow T_{z_0} S$$

*be the evaluation of global vector fields at  $z_0$ . Then, if  $\mathcal{V}$  is a covering of  $S$  such that  $H^1(S, \Theta) = H^1(\mathcal{V}, \Theta)$ ,*

- 1)  $h^0(S', \Theta') = \dim \text{Ker } e_{z_0}$ ;*
- 2) There exists a covering  $\mathcal{V}' = (V'_i)_{i \geq 0}$  of  $S'$  such that  $H^1(S', \Theta') = H^1(\mathcal{V}', \Theta')$  with the following properties:*

- i)  $V'_0$  is the inverse image by  $\Pi$  of a ball  $V_0$  centered at  $z_0$ ,*
- ii)  $V'_0$  meets only one open subset of the covering, say  $V'_1$  along a spherical shell,*

iii) For all  $i \geq 1$ , the restriction of  $\Pi$  on  $V'_i$  is an isomorphism on its image  $V_i$ ,

iv) the canonical mapping  $\Pi^* : H^1(\mathcal{V}, \Theta) \rightarrow H^1(\mathcal{V}', \Theta')$  is injective,

v) A base of  $H^1(S', \Theta')$  may be obtained from a base of  $H^1(\mathcal{V}, \Theta)$  by adding cocycles induced on  $V'_{01}$  by (at most two) non-vanishing vector fields  $Z^i$  on  $V_0$  such that the vectors  $Z^i(z_0)$  generate a supplementary subspace of  $\text{Im } e_{z_0}$  in  $T_{z_0}$ .

In particular  $h^1(S', \Theta') = h^1(S, \Theta) + \text{codim } \text{Im } e_{z_0}$ .

Proof: 1) is clear.

2) Let  $\mathcal{U} = (U_i)_{i \geq 1}$  be a locally finite covering of  $S$  such that  $H^1(S, \Theta) = H^1(\mathcal{U}, \Theta)$ . It may be supposed that  $z_0 \in U_i$  for  $i = 1, \dots, p$ . Denote by  $U'_0 \subset \subset U_0$  balls centered at  $z_0$  such that if  $i > p$ , then  $U_0 \cap U_i = \emptyset$ . Now, if  $V_0 = U_0$ ,  $V_1 = U_1 \setminus \overline{U'_0}$  and  $V_i = U_i \setminus \overline{U'_0}$ , for  $i > 1$ , there are three coverings of  $S$ ,  $\mathcal{U}$ ,  $\mathcal{U}_0 = (U_i)_{i \geq 0}$  and  $\mathcal{V} = (V_i)_{i \geq 0}$  related by the relation

$$\mathcal{V} \prec \mathcal{U}_0 \prec \mathcal{U}.$$

The canonical mappings

$$H^1(\mathcal{U}, \Theta) \rightarrow H^1(\mathcal{U}_0, \Theta) \rightarrow H^1(\mathcal{V}, \Theta)$$

are isomorphism. We define a covering  $\mathcal{V}' = (V'_i)$  of  $S'$  by  $V'_i = \Pi^{-1}(V_i)$ . The canonical morphism  $\Pi^*$  is clearly injective and the evident mapping

$$s : H^1(\mathcal{V}', \Theta') \rightarrow H^1(\mathcal{V}, \Theta)$$

is clearly surjective. Let  $\xi \notin \text{Im } e_{z_0}$  and  $\theta$  a vector field on  $U_0$  such that  $\theta(z_0) = \xi$ . Define  $\eta = (\eta_{jk}) \in Z^1(\mathcal{V}, \Theta)$  by  $\eta_{01} = \theta$  and  $\eta_{jk} = 0$  if  $\{j, k\} \neq \{0, 1\}$ . If  $\eta' = \Pi^*(\eta) \in Z^1(\mathcal{V}', \Theta')$ ,  $\eta$  and  $\eta'$  are cocycles such that  $s([\eta']) = [\eta] = 0$ , but  $[\eta'] \neq 0$ . In fact if there exist vector fields  $X'_0$  on  $V'_0$  and  $X'$  on  $S' \setminus \Pi^{-1}(U'_0)$  such that  $\theta' = X' - X'_0$  on  $V'_{01}$ , we have  $\theta = \Pi_* X' - \Pi_* X'_0$  on  $V_{01}$ . But since a vector field extends inside a ball,  $\xi = \theta(z_0) = \Pi_* X'(z_0)$  which is a contradiction. Therefore,  $\dim \text{Ker } s \geq \text{codim } \text{Im } e_{z_0}$  and it yields  $h^1(\mathcal{V}', \Theta') \geq h^1(\mathcal{V}, \Theta) + \text{codim } \text{Im } e_{z_0}$ . Now by Riemann-Roch-Hirzebruch-Atiyah-Singer theorem we have, since  $S$  and  $S'$  are VII-class surfaces,

$$h^1(S, \Theta) = h^0(S, \Theta) + 2b_2(S) \quad \text{and} \quad h^1(S', \Theta') = h^0(S', \Theta') + 2b_2(S').$$

Using  $b_2(S') = b_2(S) + 1$ , we obtain by a)

$$h^1(S', \Theta') = h^1(S, \Theta) + \text{codim } \text{Im } e_{z_0}$$

therefore

$$\begin{aligned} h^1(\mathcal{V}, \Theta) + \text{codim } \text{Im } e_{z_0} &= h^1(S, \Theta) + \text{codim } \text{Im } e_{z_0} = h^1(S', \Theta') \geq h^1(\mathcal{V}', \Theta') \\ &\geq h^1(\mathcal{V}, \Theta) + \text{codim } \text{Im } e_{z_0} \end{aligned}$$

which completes the proof.  $\square$

**Theorem 1. 13** *Let  $S$  be a surface of class  $VII_0^+$ ,  $n = b_2(S) \geq 1$  and  $\mathcal{S} \rightarrow B$  its versal deformation. Assume that there is a point  $v \in B$ ,  $\lambda_i \in \mathbb{C}$ ,  $i = 0, \dots, n-1$  such that*

$$H^0(S_v, L_{i,v} \otimes F^{\lambda_i}) \neq 0.$$

*Then there exists*

- *Holomorphic functions  $c_i : B \rightarrow \mathbb{C}$ ,*
- *Flat families of exceptional divisors  $\mathcal{C}_i$  over  $B \setminus H_i$ , where  $H_i = \{c_i = 0\}$ ,*

*such that*

- *For every  $u \in B \setminus H_i$ ,  $[C_{i,u}] = L_{i,u} \otimes F^{c_i(u)}$  and  $F^{c_i(v)} = F^{\lambda_i}$ ,*
- *$S_u$  is minimal if and only if  $u \in M := \bigcap_{i=0}^{n-1} H_i$ ,*
- *$S_u$  is a blown-up Hopf surface if and only if  $u \in B \setminus \bigcup_{i=0}^{n-1} H_i$ .*

Proof: 1) The surface  $S_v$  contains an exceptional curve of the first kind  $C$ . Changing if necessary the numbering we may suppose that  $C = C_{n-1}$  and  $L_{n-1,v} \otimes F^{\lambda_{n-1}} = [C_{n-1}]$ . By stability theorem of Kodaira [16], there exists an open neighbourhood of  $v \in B$  and a flat family  $\mathcal{C}_{n-1}$  over  $V$  of exceptional curves of the first kind.

2) Let  $pr_1 : \mathcal{S} \times \mathbb{C}^* \rightarrow \mathcal{S}$  be the first projection and define the line bundle  $\mathcal{M}_{n-1}$  over  $\mathcal{S} \times \mathbb{C}^*$  by

$$\mathcal{M}_{n-1} := pr_1^* \mathcal{L}_{n-1} \otimes_{\mathcal{O}_{\mathcal{S} \times \mathbb{C}^*}} \mathcal{F}$$

where  $\mathcal{F}$  is the tautological flat line bundle of Lemma 1.10; by  $p = \pi_{n-1} \times Id$ , the sheaf  $\mathcal{M}_{n-1}$  is flat over  $B \times \mathbb{C}^*$ . For every  $(u, \alpha) \in B \times \mathbb{C}^*$ , we have  $(\mathcal{M}_{n-1})_{|S_u \times \{\alpha\}} \simeq (\mathcal{L}_{n-1})_{|S_u} \otimes F^\alpha$ . By the semi-continuity theorem of Grauert, and because surfaces have no nonconstant meromorphic functions,

$$Z_{n-1} := \left\{ (u, \alpha) \in B \times \mathbb{C}^* \mid h^0(S_u \times \{\alpha\}, (\mathcal{M}_{n-1})_{S_u \times \{\alpha\}}) = 1 \right\}$$

is an analytic subset of  $B \times \mathbb{C}^*$  and the dimension of the intersection  $Z_{n-1} \cap V \times \mathbb{C}^*$  is  $N = \dim B$  by 1).

3) Let  $Z'_{n-1}$  be the irreducible component of  $Z_{n-1}$  such that  $\mathcal{M}_{n-1}|_{p^{-1}(Z'_{n-1} \cap V \times \mathbb{C}^*)} = [\mathcal{C}_{n-1}]$ . We have a flat family of curves

$$p : \mathcal{C}_{n-1} \rightarrow Z'_{n-1}$$

such that for  $(u, \alpha) \in Z'_{n-1}$ ,  $[C_{n-1,(u,\alpha)}] = L_{n-1,u} \otimes F^\alpha$ . For  $p = 0, 1$ , the functions

$$\begin{array}{ccc} Z'_{n-1} & \rightarrow & \mathbb{N} \\ (u, \alpha) & \mapsto & h^p(C_{n-1,(u,\alpha)}, \mathcal{O}_{C_{n-1,(u,\alpha)}}) \end{array}$$

are constant (see [5] p 96). For  $u \in V$  and  $\alpha$  such that the section of  $L_{n-1} \otimes F^\alpha$  vanishes on an exceptional curve of the first kind,

$$h^0(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 1, \quad h^1(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 0$$

therefore  $h^0(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 1$  and  $h^1(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 0$  everywhere, whence over each  $(u, \alpha) \in Z'_{n-1}$  the analytic set is connected and does not contain any elliptic curve or cycle of rational curves.

Now, we show that the intersection  $Z'_i \cap (V \times \mathbb{C}^*)$  contains only one irreducible component: In fact, if  $z = (u, \alpha)$  and  $z' = (u, \alpha')$  are two points in  $Z'_{n-1}$  over  $u \in V$ , then

$$h^0(S_u, L_{n-1,u} \otimes F^\alpha) = h^0(S_u, L_{n-1,u} \otimes F^{\alpha'}) = 1,$$

whence  $F^{\alpha/\alpha'}$  has a meromorphic section and by [20] (2.10),  $F^{\alpha/\alpha'} = [D]$  with  $D = mE + nF$ , where  $E, F$  are elliptic curves or cycles of rational curves such that  $E^2 = F^2 = 0$ . It means that for  $L_{n-1,u} \otimes F^\alpha = [C_{n-1,u}]$  and  $L_{n-1,u} \otimes F^{\alpha'} = [C'_{n-1,u}]$  we have

$$[C_{n-1,u}] = L_{n-1,u} \otimes F^\alpha = L_{n-1,u} \otimes F^{\alpha'} \otimes F^{\alpha/\alpha'} = [C'_{n-1,u} + D]$$

and  $C_{n-1,u}$  would not be connected, a contradiction.

As consequence  $Z'_i$  cannot accumulate on

$$B \times \{0\} \cap V \times \mathbb{P}^1(\mathbb{C}), \quad \text{or} \quad B \times \{\infty\} \cap V \times \mathbb{P}^1(\mathbb{C}).$$

By Remmert-Stein theorem the closure

$$G_{n-1} = \overline{Z'_{n-1}} \subset B \times \mathbb{P}^1(\mathbb{C})$$

is an irreducible analytic set of codimension one. The restriction

$$pr_1 : G_{n-1} \rightarrow B$$

is proper; therefore,  $pr_1(G_{n-1})$  is an analytic subset of  $B$ . Since it contains the open set  $V$ ,  $pr_1(G_{n-1}) = B$ . Now,  $(G_{n-1}, pr_1, B)$  is a ramified covering which has only one sheet over  $V$ , hence  $G_{n-1}$  is the graph of a holomorphic mapping  $c_{n-1} : B \rightarrow \mathbb{P}^1(\mathbb{C})$ . Define  $H_{n-1} := c_{n-1}^{-1}(0) \cup c_{n-1}^{-1}(\infty)$ . So far, we have considered the family of curves  $\mathcal{C}_{n-1}$  over  $Z'_{n-1}$ . From now on, we shall consider it over  $B_{n-1} := B \setminus H_{n-1}$ .

4) If for  $i \neq n-1$ ,  $L_{i,v} \otimes F^{\lambda_i} = [C_{i,v}]$  with  $C_{i,v}$  reducible, it means that  $C_i = \pi^*(C'_i)$  where  $\pi$  is a finite sequence of blowing-ups and  $C'_i$  is an exceptional curve of the first kind. By lemma (12), there is a deformation in which the blown-up points are moved outside  $C'_i$ . Therefore changing the point  $v$  in  $V' \subset V$  we may suppose that  $C_{i,v}$  is an exceptional curve of the first kind and apply 3).

5) We show now that for all  $u \in B' := B \setminus H_i$ ,  $S_u$  is a blown-up Hopf surface: By Iitaka theorem we may blow down the exceptional divisors  $\mathcal{C}_{i|B'}$  over  $B'$ . Let  $p : \mathcal{S} \rightarrow \mathcal{S}'$  be the canonical mapping and  $\Pi' : \mathcal{S}' \rightarrow B'$  the induced family. Using classification of complex surfaces of class VII<sub>0</sub> with  $b_2(\mathcal{S}) = 0$  (see [18]),  $S'_u$  is a Hopf surface or a Inoue surface [15] of type  $S_M$ ,  $S_{N,p,q,r;t}^{(+)}$  or  $S_{N,p,q,r}^{(-)}$ . We have to exclude the last three types: By [11] we may suppose that  $\mathcal{S}$  has no non-trivial global vector field, therefore  $N = h^1(\mathcal{S}, \Theta) = 2n$  and restricting if necessary  $B$  we have  $h^1(S_u, \Theta_u) = 2n$  for all  $u \in B$ . We denote

by  $\mathcal{K}'$  the relative canonical bundle over  $\mathcal{S}'$ . For every  $u \in B'$  there exists a unique  $f(u) \in \mathbb{C}^*$  such that  $-K'_u = F^{f(u)}$ ; by Grauert theorem,  $f : B' \rightarrow \mathbb{C}^*$  is holomorphic. The relative canonical bundle  $\mathcal{K}$  over  $\mathcal{S}$  satisfies

$$\mathcal{K} = p^* \mathcal{K}' \otimes \bigotimes_{i=0}^{n-1} [\mathcal{C}_i] = F^{f^{-1}} \otimes \bigotimes_{i=0}^{n-1} [\mathcal{C}_i].$$

If there exists a point  $u \in B'$  such that  $S'_u$  is of type  $S_{N,p,q,r;t}^{(+)}$  or  $S_{N,p,q,r}^{(-)}$ , then in a neighbourhood of  $u$ , all surfaces have the same type by [15] and by the theorem of Bombieri [15] p280, the function  $f$  is real valued hence constant, in particular globally defined. Therefore  $\bigotimes_{i=0}^{n-1} [\mathcal{C}_i] = F^f \otimes \mathcal{K}$  extends over  $B$ , which is impossible. Remains the case  $S_M$  which is rigid: all the surfaces  $S'_u$  are isomorphic hence by (11) Example 3,  $-K'_u = F^{\alpha\beta}$ , therefore as before  $\bigotimes_i [\mathcal{C}_i]$  should extend which is impossible.

6) The functions  $c_i$  are  $\mathbb{C}$ -valued: In fact, suppose that  $c_i^{-1}(\infty) \neq \emptyset$ ; let  $A$  be an irreducible component of  $c_i^{-1}(\infty)$ . Since  $c$  has values in  $\mathbb{C}$ , there exists an index  $j$  such that  $c_j$  vanishes along  $A$ . Let  $\mu$  (resp.  $\nu$ ) be the order of the pole (resp. zero) of  $c_i$  (resp.  $c_j$ ) at a point  $a \in A$ . Then on a disc  $\Delta$  containing  $a$ ,  $g := c_i^\nu c_j^\mu \in \mathcal{O}^*(\Delta)$  and the family of positive divisors

$$[\nu \mathcal{C}_i + \mu \mathcal{C}_j] = \mathcal{L}_i^\nu \otimes \mathcal{L}_j^\mu \otimes F^g$$

extends on  $\Delta$ . But it implies that  $\mathcal{C}_i$  and  $\mathcal{C}_j$  extend also, which yields a contradiction.  $\square$

**Definition 1. 14** *Let  $S$  be a surface of class  $VII_0^+$ . We shall say that  $S$  is a degeneration of blown-up primary Hopf surfaces if there is a deformation  $\mathcal{S} \rightarrow \Delta$  over the unit disc of  $S \simeq S_0$ , such that  $S_u$  is a blown-up primary Hopf surface for  $u \neq 0$ .*

If  $\pi_1(S) = \mathbb{Z}$ , a surface which can be deformed into a  $n$  times blown-up surface, is a degeneration of blown-up primary Hopf surfaces. The surface  $S_u$  is defined by a contraction

$$F_u(z) = (\alpha_1(u)z_1 + s(u)z_2^m, \alpha_2(u)z_2)$$

$$\text{with } 0 < |\alpha_1(u)| \leq |\alpha_2(u)| < 1 \text{ and } s(u)(\alpha_2^m(u) - \alpha_1(u)) = 0$$

or

$$F_u(z) = (\alpha_1(u)z_1, \alpha_2(u)z_2 + s(u)z_1^m)$$

$$\text{with } 0 < |\alpha_2(u)| \leq |\alpha_1(u)| < 1 \text{ and } s(u)(\alpha_1^m(u) - \alpha_2(u)) = 0.$$

In both cases there is at least one elliptic curve  $E_2$  (resp.  $E_1$ ) induced by  $\{z_2 = 0\}$  (resp.  $\{z_1 = 0\}$ ) and another  $E_1$  (resp.  $E_2$ ) induced by  $\{z_1 = 0\}$  (resp.  $\{z_2 = 0\}$ ) if  $s = 0$ . The trace

$$t(u) = \text{tr}(S'_u) = \text{tr}(S_u) = \text{tr}(DF_u(0)) = \alpha_1(u) + \alpha_2(u)$$

and the determinant

$$d(u) = \det DF_u(0) = \alpha_1(u)\alpha_2(u)$$

are bounded holomorphic functions on  $B'$  hence extend on  $B$ . They depend only on the conjugation class of  $F_u$ . We call  $tr(u)$  the **trace of the surface**  $S_u$ .

By [17] II p696, we have the following description of the canonical bundle  $K'_u$  of  $S'_u$ : If  $S'_u$  is a diagonal Hopf surface then

$$K'_u = [-E_{1,u} - E_{2,u}] = F^{(\alpha_1(u)\alpha_2(u))^{-1}}.$$

If  $s(u) \neq 0$ , i.e.  $S$  is not diagonal, then since  $\alpha_2^m(u) = \alpha_1(u)$ ,

$$K'_u = [-(m+1)E_{2,u}] = F^{\alpha_2(u)^{-(m+1)}} = F^{(\alpha_1(u)\alpha_2(u))^{-1}}.$$

By [5] I.9.1 (vii), the canonical bundle  $K_u$  of  $S_u$  over  $B'$  satisfies

$$K_u = p^* K'_u \otimes \bigotimes_{0 \leq i \leq n-1} [C_{i,u}].$$

If  $\mathcal{K}$  denotes the relative canonical bundle and  $F$  denotes the tautological flat line bundle,

$$\mathcal{K} = F^{(\alpha_1\alpha_2)^{-1}} \otimes \bigotimes_i [\mathcal{C}_i] = F^f \otimes \bigotimes_i \mathcal{L}_i$$

where

$$f := \frac{\prod_i c_i}{\alpha_1\alpha_2}.$$

Since  $\mathcal{K}$  and  $\mathcal{L}_i$  are globally defined on  $B$ ,  $F^f$  is globally defined on  $B$  and  $f \in \mathcal{O}^*(B)$ . Twisting, for example,  $\mathcal{L}_1$  by  $F^{f^{-1}}$ , we may suppose that  $f = 1$  and then

$$c = \prod_{i=0}^{n-1} c_i = \alpha_1\alpha_2 = d \in \mathcal{O}^*(B)$$

satisfies  $\|c\|_\infty \leq 1$ .

For contracting germs associated to surfaces containing GSS we refer to [6].

**Proposition 1. 15** *Let  $S$  be a surface obtained by degeneration of blown-up minimal Hopf surfaces. If  $tr(S) \neq 0$ , then  $S$  contains a GSS and if  $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  is a contracting germ of mappings associated to  $S$ ,  $tr(S) = trDF(0)$ .*

Proof: One of the two functions  $\alpha_i$ ,  $i = 1, 2$ , say  $\alpha_2$ , admit a limit  $\alpha_2(0) \neq 0$ , and  $\alpha_1(0) = 0$ . Therefore  $F^{\alpha_2}$  is globally defined. Since  $|\alpha_1(u)| < |\alpha_2(u)|$ , the elliptic curve  $E_{2,u}$  induced by  $z_2 = 0$  exists for all  $u \notin H$ , hence  $[E_{2,u}] = F^{\alpha_2}$ . Since  $u \mapsto h^0(S_u, F^{\alpha_2(u)})$  is upper-semicontinuous we obtain in  $S$  a flat cycle of rational curves  $E_{2,0}$ . By Enoki theorem,  $S$  contains a GSS.  $\square$

## 2 Surfaces with a cycle of rational curves

**Definition 2. 16** *Let  $S$  be a surface and  $C$  be an analytic subset of dimension one. We say that  $C$  is a  $r$ -cycle of rational curves if*

- $C$  is an elliptic curve when  $r = 0$ ,
- $C$  is a rational curve with a double point when  $r = 1$ , and
- $C = D_0 + \cdots + D_{r-1}$ , is a reduced effective divisor such that  $D_i$  is a non-singular rational curve for  $i = 0, \dots, r-1$  and  $D_0 D_1 = \cdots = D_{r-2} D_{r-1} = D_{r-1} D_0 = 1$ ,  $D_i D_j = 0$  in all other cases, when  $r \geq 2$ .

We denote by  $\sharp(C) = r \geq 0$  the number of rational curves of the cycle  $C$ .

With notations of theorem (8), we have

**Theorem 2. 17 ([21] (1.7))** *Let  $S$  be a  $VII_0^+$  surface with  $n = b_2(S)$ . Assume that  $S$  contains exactly one cycle  $C$  of rational curves such that  $C^2 < 0$ . Then*

- $C \sim -(L_r + \cdots + L_{n-1})$  for some  $1 \leq r \leq n-1$ , if  $S$  is not an odd Inoue-Hirzebruch surface, or
- $C \sim -(L_0 + \cdots + L_{n-1}) + F_2$ , with  $F_2$  of order two, if  $S$  is an odd Inoue-Hirzebruch surface.

**Remark 2. 18** *If  $S$  is an even Inoue-Hirzebruch surface (=hyperbolic I-H surface), the two cycles  $C$  and  $C'$  satisfy*

$$C \sim -(L_r + \cdots + L_{n-1}), \quad C' \sim -(L_0 + \cdots + L_{r-1}).$$

The following lemma plays a crucial role in the computation of self-intersection of the cycle  $C$ .

**Lemma 2. 19 ([21] (2.4))** *Let  $S$  be a surface of class  $VII_0^+$  and without divisor  $D$  such that  $D^2 = 0$ . Let  $L_I := \sum_{i \in I} L_i$ ,  $L = L_I + F$ ,  $F \in H^1(S, \mathcal{O}_S^*)$  for a nonempty subset  $I \subset [0, n-1]$ . Then we have:*

- i) *If  $I \neq [0, n-1]$ , then  $H^q(S, L) = 0$  for any  $q$ .*
- ii) *If  $L \otimes \mathcal{O}_C = \mathcal{O}_C$ , then  $I = [0, r-1]$ , and  $F = \mathcal{O}_S$ ,  $K_S - L + C = \mathcal{O}_S$ .*
- iii) *If  $LC_i = 0$  for any irreducible component  $C_i$  of  $C$ , then  $I = [0, r-1]$ .*

### 2.1 Surfaces with numerically $m$ -anticanonical divisor

**Definition 2. 20** *Let  $S$  be a compact complex surface of the  $VII_0$  class with  $b_2(S) > 0$  and let  $m \geq 0$  be an integer. We shall say that  $S$  admits a **numerically  $m$ -anticanonical divisor** if there exists a divisor  $D_m$ , and a flat line bundle  $F$  in  $H^1(S, \mathbb{C}^*)$  such that in  $H^1(S, \mathcal{O}^*)$ ,*

$$mK + F + [D_m] = 0.$$

We shall say that  $S$  admits a numerically anticanonical divisor, shortly a **NAC divisor**, if there exists an integer  $m$  such that there exists a numerically  $m$ -anticanonical divisor.

**Remark 2. 21** Let  $S$  be a compact complex surface of the  $VII_0$  class with  $b_2(S) > 0$ . Let  $m \geq 1$  be an integer and  $F \in H^1(S, \mathbb{C}^*)$ . Then  $H^0(S, mK \otimes F) = 0$ . In fact, a section of  $mK \otimes F$  must vanish; let  $[\Delta] = mK \otimes F$  be the associated divisor. Then

$$0 = (mK + F - \Delta)\Delta = mK\Delta - \Delta^2.$$

Since  $mK\Delta \geq 0$  and  $\Delta^2 \leq 0$ ,  $\Delta^2 = 0$ , therefore  $b_2(S) = -K^2 = 0$ . This means that there is no numerically m-canonical divisors.

For the convenience of the reader we recall basic known facts with slightly different proofs (see [21] lemma (3.1))

**Lemma 2. 22** 1) If  $D_m$  exists, it is a positive divisor, hence  $H^0(S, -mK - F) \neq 0$ .

2) If an irreducible curve  $C$  meets the support  $|D_m|$  of  $D_m$  then  $C$  is contained in  $|D_m|$ .

Proof: 1) We denote by  $D_i$  the irreducible components of  $D_m$ . Let  $D_m = \sum_i k_i D_i = A - B$  where  $A = \sum_{i|k_i > 0} k_i D_i \geq 0$  and  $B = \sum_{i|k_i < 0} (-k_i) D_i \geq 0$  have no common component. If  $B \neq 0$ ,  $B^2 = \sum (-k_i) B D_i < 0$ , there exists an index  $j$  such that  $B D_j < 0$ . Therefore,

$$0 \leq mK D_j = (-F - D_m) D_j = -D_m D_j = -A D_j + B D_j < 0$$

... a contradiction.

2) If  $C$  is an irreducible curve and meets  $|D_m|$ ,  $D_m C = -mK C \leq 0$ , therefore  $C$  is contained in  $|D_m|$ .  $\square$

If there exists a non trivial divisor  $A$  such that  $A^2 = 0$ , then a numerically m-anticanonical divisor exists if and only if  $S$  is a Inoue surface. In this case,  $S$  contains an elliptic curve  $E$ , a cycle  $\Gamma$  of rational curves,  $m = 1$  and  $K + E + \Gamma = 0$ . When there is no non-trivial flat divisor, the numerically m-anticanonical divisor  $D_m$  is clearly unique.

## 2.2 The reduction lemma

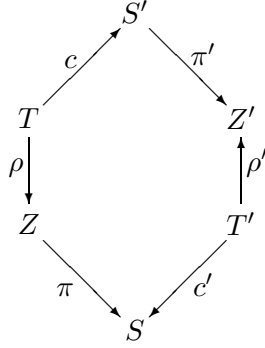
**Definition 2. 23** The least integer  $m$  such that there exists a numerically m-anticanonical divisor, is called the **index** of the surface  $S$  and will be denoted by  $m(S)$ .

The index  $m(S)$  of the surface  $S$  is the lcm of the denominators of the coefficients  $k_i$  of  $D_1 = \sum_i k_i D_i$ .

The proof of [12] (1.3) works under the following relaxed hypothesis:

**Lemma 2. 24 (Reduction lemma)** Let  $S$  be a surface of class  $VII_0$  with  $b_2(S) > 0$  and index  $m = m(S) > 1$ . Then there exists a diagram





where

- i)  $(Z, \pi, S)$  is a  $m$ -fold ramified covering space of  $S$ , branched over  $D_m$ , endowed with an automorphism group isomorphic to  $\mathbb{U}_m$  which acts transitively on the fibers,
- ii)  $(T, \rho, Z)$  is the minimal desingularization of  $Z$ ,
- iii)  $(T, c, S')$  is the contraction of the (possible) exceptional curves of the first kind,
- iv)  $S'$  is a surface of class  $VII_0$ , with  $b_2(S') > 0$ , with action of  $\mathbb{U}_m$ , with index  $m(S') = 1$ ,
- v)  $(S', \pi', Z')$  is the quotient space of  $S'$  by  $\mathbb{U}_m$ ,
- vi)  $(T', \rho', Z')$  the minimal desingularization of  $Z'$ ,
- vii)  $(T', c', S)$  is the contraction of the (possible) exceptional curves of the first kind,

such that the restriction over  $S \setminus D$  is commutative, i.e.

$$\theta := \pi \circ \rho \circ c^{-1} = c' \circ \rho'^{-1} \circ \pi' : S' \setminus D' \rightarrow S \setminus D$$

and  $(S' \setminus D', S \setminus D)$  is a  $m$ -fold non ramified covering. Moreover

- $S$  contains a GSS if and only if  $S'$  contains a GSS,  $S$  and
- The maximal divisors  $D$  and  $D'$  of  $S$  and  $S'$  respectively have the same number of cycles and branches.

**Corollary 2. 25** *If  $S$  admits a NAC divisor  $D_m$ , then the support of  $D_m$  contains a cycle.*

Proof: We may suppose that there is no divisor such that  $D^2 = 0$ . By reduction lemma it is sufficient to prove that the support of  $D_{-K}$  contains a cycle. Suppose that there exists a divisor  $D_{-K}$  and a flat line bundle  $F$  such that  $K + D_{-K} + F = 0$ . By Cartan-Serre duality

$$h^2(S, \mathcal{O}_S(-D_{-K})) = h^2(S, K + F) = h^0(S, -F) = \begin{cases} 0 & \text{if } F \neq 0 \\ 1 & \text{if } F = 0 \end{cases}$$

By Riemann-Roch formula,

$$h^0(S, -F) - h^1(S, -F) = h^0(S, -F) - h^1(S, -F) + h^0(S, K + F) = \chi(F) = 0,$$

hence

$$h^1(S, -F) = \begin{cases} 0 & \text{if } F \neq 0 \\ 1 & \text{if } F = 0 \end{cases}$$

We have

$$0 \rightarrow \mathcal{O}_S(-D_{-K}) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{D_{-K}} \rightarrow 0$$

- If  $F \neq 0$ , the long exact sequence yields

$$h^0(S, \mathcal{O}_{D_{-K}}) = h^1(S, \mathcal{O}_{D_{-K}}) = 1.$$

The support of  $D_{-K}$  is connected and by [20] (2.7),  $h^1(S, \mathcal{O}_{(D_{-K})_{red}}) \geq 1$  hence  $(D_{-K})_{red}$  contains a cycle of rational curves.

- If  $F = 0$ , the associated long exact sequence and [20] (2.2.1) imply

$$1 \leq h^0(S, \mathcal{O}_{D_{-K}}) = h^1(S, \mathcal{O}_{D_{-K}}) \leq 2.$$

As before  $h^1(S, \mathcal{O}_{(D_{-K})_{red}}) \geq 1$  and there is at least one cycle.

If  $h^1(S, \mathcal{O}_{(D_{-K})_{red}}) = 2$ , then by the already quoted result (2.2.1) there are two cycles of rational curves.

□

### 2.3 Characterization of Inoue-Hirzebruch surfaces

**Lemma 2. 26** *Let  $S$  be a surface with a NAC divisor  $D_{-K} = \sum k_i D_i$ . We suppose that the maximal divisor contains a cycle of rational curves  $C = D_0 + \dots + D_{s-1}$  with  $s \geq 1$  irreducible curves. If there exists  $j \leq s-1$  such that  $k_j = 1$ , then  $k_i = 1$  for all  $i = 1, \dots, s-1$  and  $C$  has no branch.*

**Proof:** **Case  $s = 1$ :** Since  $D_0$  is a rational curve with a double point, the adjunction formula yields

$$D_0^2 = -K D_0 = D_{-K} D_0 = \sum k_i D_i D_0 = D_0^2 + \sum_{i>0} k_i D_i D_0.$$

hence  $\sum_{i>0} k_i D_i D_0 = 0$  and  $C$  has no branch.

**Case  $s \geq 2$ :** By adjunction formula,

$$2 + D_j^2 = -K D_j = \sum_i k_i D_i D_j = D_j^2 + \sum_{i \neq j} k_i D_i D_j,$$

whence  $2 = \sum_{i \neq j} k_i D_i D_j$ . Since  $\sum_{i \neq j} D_i D_j \geq 2$ ,  $D_j$  meets at most two curves  $D_{j-1}$  and  $D_{j+1}$  (one if the cycle contains two curves) and  $k_{j-1} = k_{j+1} = 1$ . By connectivity we conclude. □

**Proposition 2. 27** *Let  $S$  be a surface with a NAC divisor  $D_{-K} = \sum k_i D_i$  and let  $C = D_0 + \dots + D_{s-1}$  be a cycle contained in the support of  $D_{-K}$ . One of the following conditions holds:*

i) There exists an index  $0 \leq j \leq s-1$  such that  $k_j = 1$ , then  $S$  is a Inoue surface or a Inoue-Hirzebruch surface.

ii) For every  $0 \leq j \leq s-1$ ,  $k_j \geq 2$ , then  $C$  has at least one branch and the support  $D = |D_{-K}|$  of  $D_{-K}$  is connected. More precisely, if  $k = \max\{k_i \mid 0 \leq i \leq s-1\}$ , there exists a curve  $D_j$  and a branch  $H_j > 0$  such that  $k_j = k$  and  $D_j H_j > 0$ .

In particular, each connected component of  $|D_K|$  contains a cycle.

**Definition 2. 28** With preceeding notations, a curve  $D_j$  in the cycle such that  $D_j H_j > 0$  will be called the root of the branch  $H$ .

Proof of (27): Taking if necessary a double covering it may be supposed, by [20](2.14), that the cycle has at least two curves, hence all the curves are regular. Let

$$D_{-K} = A + B = \sum_{i=0}^{p-1} k_i D + \sum_{i=p}^{p+q} k_i D_i$$

where the support of  $A = \sum_{i=0}^{p-1} k_i D_i$  is the connected component of the cycle  $C = \sum_{i=0}^{s-1} D_i$ .

If  $B$  contains another cycle resp. an elliptic curve), then  $S$  is a Inoue-Hirzebruch surface [20] (8.1) (resp. a Inoue surface [20] (10.2)); suppose therefore that the support  $|B|$  of  $B$  is simply-connected. We have to show that  $|D_{-K}|$  is connected: In fact, let  $B_0$  be a connected component of  $|B|$ . There is a proper mapping  $p : S \rightarrow S'$  onto a normal surface  $S'$  with normal singularities  $a = p(|A|)$  and  $b = p(B_0)$ . Since  $B$  is simply connected,  $F$  is trivial on a strictly pseudo-convex neighbourhood  $U$  of  $B$  and thus a holomorphic section of  $-K - F$  yields a non vanishing holomorphic 2-form on  $U \setminus B$ , i.e.  $(S', b)$  is Gorenstein. If  $(S', b)$  would be an elliptic singularity, a two-fold covering  $T$  of  $S$  should contain three exceptional connected divisors such that their contractions  $q : T \rightarrow T'$  would fullfil  $h^0(T', R^1 q_* \mathcal{O}_T) = 3$ . However, by Leray spectral sequence, there is an exact sequence

$$(\dagger) \quad 0 \rightarrow H^1(T', \mathcal{O}_{T'}) \rightarrow H^1(T, \mathcal{O}_T) \rightarrow H^0(T', R^1 q_* \mathcal{O}_T) \rightarrow H^2(T', \mathcal{O}_{T'}) \rightarrow H^2(T, \mathcal{O}_S)$$

where

$$p_g = h^2(T, \mathcal{O}_T) = 0, \quad q = h^1(T, \mathcal{O}_T) = 1 \quad \text{and} \quad h^2(T', \mathcal{O}_{T'}) = h^0(T', \omega_{T'}) \leq 1,$$

by Serre-Grothendieck duality and because  $T'$  has no non-constant meromorphic functions. By  $(\dagger)$  we obtain a contradiction, hence  $(S', b)$  is a Gorenstein rational singularity, hence a Du Val singularity. However, such a singularity has a trivial canonical divisor, therefore  $B = 0$  and  $D_{-K} = A$ .

i) Suppose that there is an index  $j \leq s-1$  such that  $k_j = 1$ . By lemma (26),  $C$  has no branch and  $s = p$ . By adjunction formula, we have for every  $0 \leq i \leq s-1$ ,

$$D_i^2 + 2 = -K D_i = D_{-K} D_i = k_{i-1} + k_{i+1} + k_i D_i^2,$$

therefore

$$(\star) \quad (k_{i-1} - 1) + (k_{i+1} - 1) = (k_i - 1)(-D_i^2).$$

By  $(\star)$ ,  $k_i = 1$  for all  $0 \leq i \leq p - 1$ .

- If for every  $0 \leq i \leq s - 1$ ,  $D_i^2 = -2$ ,  $S$  is a Enoki surface by [14]. Moreover, by hypothesis,  $S$  admits a NAC divisor, hence is a Inoue surface or
- There is at least one index  $k \leq s - 1$  such that  $D_k^2 \leq -3$  whence  $C^2 < 0$ . If there is no other cycle, we have shown that  $D_{-K} = C$ , hence

$$-b_2(S) = K^2 = C^2,$$

$S$  is a Inoue-Hirzebruch surface by [20] (9.2).

In case ii), if the maximum  $k$  is reached at a curve  $D_i$ ,  $i \leq s - 1$ , which is not a root, then applying again  $(\star)$ , we obtain  $k_{i-1} = k_{i+1} = k_i = k$ . By connexity, we reach a root.  $\square$

**Proposition 2. 29** *Let  $S$  be a surface of class  $VII_0^+$  admitting a NAC divisor  $D_m$ . Then one of the following conditions is fulfilled:*

- The maximal divisor  $D$  is connected and contains a cycle with at least one branch;*
- $S$  is a Inoue surface or a (even or odd) Inoue-Hirzebruch surface.*

Proof: By (25)  $S$  contains at least one cycle By (24)  $S'$  admits a NAC divisor  $D_{-K}$ . Then by (27), and (24)  $S'$  and  $S$  are of the same type.  $\square$

## 2.4 Surfaces with singular rational curve

**Theorem 2. 30** *Let  $S$  be a surface with  $n = b_2(S)$  containing a singular rational curve  $D_0$  with a double point. Then*

1)  $D_0 \sim -(L_1 + \cdots + L_{n-1})$  and the connected component containing  $D_0$  is  $D_0 + D_1 + \cdots + D_p$  for  $1 \leq p \leq n - 1$ ,  $D_i \sim L_i - L_{i-1}$ ,  $1 \leq i \leq p$ . In particular  $D_0^2 = -(n - 1)$ ,  $D_1^2 = \cdots = D_p^2 = -2$ .

2) *The following conditions are equivalent:*

- There exists an integer  $m \geq 1$  such that there exists a numerically  $m$ -anticanonical divisor,*
- $S$  contains a GSS.*

*When these conditions are fulfilled then  $S$  is either a Inoue-Hirzebruch surface and  $m(S) = 1$  or  $D_0$  has a branch and its index satisfies  $m(S) = n - 1$ .*

Proof: 1) By (17), there exists  $r \geq 1$  such that  $D_0 = -(L_r + \cdots + L_{n-1})$ . We have  $L_0 D_0 = 0$ , hence  $r = 1$  by (19) iii). We show by induction on  $i \geq 1$  that if there is a non singular rational curve  $D_i$  such that  $(D_0 + \cdots + D_{i-1})D_i \neq 0$  then  $D_i$  is unique and  $D_i = L_i - L_{i-1}$ . By [20] (2.2.4), and unicity  $D_0 D_i = \cdots = D_{i-2} D_i = 0$ ,  $D_{i-1} D_i = 1$  if  $i \geq 2$ .

Suppose that there is a curve, say  $D_1$  such that  $D_0 D_1 = 1$ . By [21] (2.6) (see lemma 32 below) we set  $D_1 \sim L_1 - L_{I_1}$ , then

$$1 = D_0 D_1 = -(L_1 + \cdots + L_{n-1})(L_1 - L_{I_1}) = 1 + (L_1 + \cdots + L_{n-1})L_{I_1},$$

hence  $I_1 = \{0\}$  and  $D_1 \sim L_1 - L_0$ . If there were another curve, say  $D_2$  meeting  $D_0$ , the same argument shows that  $D_2 = L_2 - L_0$ , but in this case  $D_1 D_2 = -1$  and it is impossible. It shows the unicity. Suppose that for  $i > 1$ ,  $D_i$  exists and  $D_i \sim L_i - L_{I_i}$ . We have the equations

$$\begin{array}{rclclcl} 0 & = & D_0 D_i & = & -(L_1 + \cdots + L_{n-1})(L_i - L_{I_i}) & = & 1 + (L_1 + \cdots + L_{n-1})L_{I_i} \\ 0 & = & D_1 D_i & = & -(L_1 - L_0)(L_i - L_{I_i}) & = & -L_1 L_{I_i} + L_0 L_{I_i} \\ & & \vdots & & & & \vdots \\ 0 & = & D_{i-2} D_i & = & -(L_{i-2} - L_{i-3})(L_i - L_{I_i}) & = & -L_{i-2} L_{I_i} + L_{i-3} L_{I_i} \\ 1 & = & D_{i-1} D_i & = & -(L_{i-1} - L_{i-2})(L_i - L_{I_i}) & = & -L_{i-1} L_{I_i} + L_{i-2} L_{I_i} \end{array}$$

which yield from the last to the second equation

$$i-1 \in I_i, \quad i-2 \notin I_i, \quad \dots, \quad 0 \notin I_i.$$

The first one implies that  $I_i$  contains exactly one index, hence

$$D_i \sim L_i - L_{i-1}.$$

We prove unicity as for  $i = 1$ .

2) By 1) the intersection matrix of the connected component of the cycle is the  $(p+1, p+1)$  matrix,  $p \geq 0$ ,

$$M = \begin{pmatrix} -(n-1) & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & & \vdots \\ 0 & 1 & -2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}$$

and it is easy to check that

$$\det M = (-1)^{p+1} [(n-1)(p+1) - p].$$

A numerically  $m$ -anticanonical divisor  $D_m = \sum_{i=0}^p k_i D_i$  supported by the connected component containing the cycle satisfies the linear system

$$M \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_p \end{pmatrix} = \begin{pmatrix} -m(n-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore

$$(*) \quad k_i = \frac{m(n-1)(p+1-i)}{(n-1)(p+1)-p}, \quad \text{for every } 0 \leq i \leq p.$$

Notice that

$$k_i = (p+1-i)k_p \quad \text{for every } 0 \leq i \leq p,$$

therefore all  $k_i$  are integers if and only if  $k_p$  is an integer. By assumption

$$D_m^2 = (-mK)^2 = -m^2n,$$

and we have  $D_m D_i = -mK D_i = 0$  for  $i = 1, \dots, p$ . We have to consider two cases:

**First case:**  $p \geq 1$  then

$$-m^2n = \sum_{i=0}^p k_i D_i D_m = k_0 D_0 D_m = k_0^2 D_0^2 + k_0 k_1 = -\frac{m^2(n-1)^2(p+1)}{(n-1)(p+1)-p}$$

and this condition is equivalent to

$$p = n - 1.$$

Hence  $S$  has a numerically  $m$ -anticanonical divisor if and only if  $S$  contains  $n$  rational curves if and only if  $S$  contains a GSS by [12].

Replacing in (\*) we obtain

$$k_i = \frac{m(n-i)}{n-1}, \quad \text{for every } 0 \leq i \leq n-1.$$

Hence we conclude

$$m(S) = n - 1.$$

**Second case:**  $p = 0$ , then

$$-m^2n = k_0^2 D_0^2 = -k_0^2(n-1)$$

However, this equation has no solution in integers, therefore the support of  $D_m$  is not connected. We conclude by (29).  $\square$

**Corollary 2. 31** *If  $S$  in the  $VII_0^+$  class has a NAC divisor and contains a rational curve with a double point, then  $S$  contains a GSS.*

## 2.5 Cycles with at least two rational curves

**Lemma 2. 32** ([21] (2.5), (2.6), (2.7)) *1) Let  $D$  be a nonsingular rational curve. If  $D \sim a_0 L_0 + \dots + a_{n-1} L_{n-1}$ , then there exists a unique  $i \in [0, n-1]$  such that  $a_i = 1$  or  $-2$ , and  $a_j = 0$  or  $-1$  for  $j \neq i$ .*

*2) If  $D$  is not contained in the cycle then  $D \sim L_i - L_I$  for some  $i \in [0, n-1]$  and  $I \subset [0, n-1]$ ,  $i \notin I$ .*

*3) Let  $D_1$  and  $D_2$  two distinct nonsingular rational curves such that  $D_1 \sim L_{i_1} - L_{I_1}$ ,  $D_2 \sim L_{i_2} - L_{I_2}$ , then  $i_1 \neq i_2$ .*

**Proposition 2. 33** *Let  $S$  be a surface containing a cycle  $C = D_0 + \dots + D_{s-1}$  of  $s \geq 2$  nonsingular rational curves. Suppose that there exists a rational curve  $E$  such that  $E.C = 1$ , and let  $D = \sum_i D_i$  be the maximal connected divisor containing  $C$ . Then every curve  $D_i$  is of the type  $D_i \sim L_i - L_I$  for  $i \notin I$ .*

Proof: By (32), every curve  $D_i$  is of type  $D_i \sim L_i - L_I$  (type **a**) or  $D_i \sim -2L_i - L_I$  (type **b**). Suppose that  $C$  contains at least one curve of type **b**. Since  $(-2L_i - L_I)(-2L_j - L_J) \leq 0$  two such curves cannot meet, in particular if  $s = 2$ ,  $C$  contains at most one curve of type **b**.

Suppose  $s \geq 3$ . If  $D_i$  of type **b**,  $D_i$  meets curves  $D_j$  of type **a**. We have

$$1 = (-2L_i - L_I)(L_j - L_J) = -2L_iL_j - L_jL_I + 2L_iL_J + L_IL_J$$

and

- either  $j \in I$ ,  $i \neq j$ ,  $i \notin J$ ,  $I \cap J = \emptyset$ , whence

$$D_i + D_j \sim -2L_i - L_I + L_j - L_J = -2L_i - L_{I \cup J \setminus \{j\}}.$$

- or  $j \notin I$ ,  $i = j$ ,  $i \notin J$  and  $I \cap J$  contains one element, say  $k$ . Setting  $I' = I \setminus \{k\}$  and  $J' = J \setminus \{k\}$  we obtain

$$D_i + D_j \sim -2L_k - L_{\{i\} \cup I' \cup J'}.$$

By ([21] (1.7)), there is a proper smooth family of compact surfaces  $\pi : \mathcal{S} \rightarrow \Delta$  over the unit disc, a flat divisor  $\mathcal{C}$  such that  $C_0 = C$  and for  $u \neq 0$ ,  $C_u \sim D' + (C - D_i - D_j)$ , with  $D' \sim D_i + D_j$  of type **b**, in particular  $\sharp(C_u) = \sharp(C) - 1$ . Therefore, repeating if necessary such a deformation we obtain a contradiction if  $C$  contains two curves of type **b**. Now if  $C$  contains a curve of type **b**, we take a double covering  $p : S' \rightarrow S$  of  $S$ . The surface  $S'$  of type VII<sub>0</sub> satisfies  $b_2(S') = 2b_2(S)$  and by [20] (2.14) contains a cycle of  $2s$  rational curves. Applying the same arguments to  $S'$ ,  $p^*D_i \sim -2p^*L_i - p^*L_I$  is the union of two rational curves of type **b**, hence a contradiction. Finally, there is no curve of type **b**.  $\square$

**Lemma 2. 34** *Let  $S$  be a surface containing a cycle  $C = D_0 + \dots + D_{s-1}$  of  $s \geq 2$  nonsingular rational curves and  $D = \sum_{i=0}^p D_i$  the maximal connected divisor containing  $C$ .*

- 1) *Let  $D_j$ ,  $D_k$  and  $D_l$  three distinct nonsingular rational curves,  $D_j \sim L_{i_j} - L_{I_j}$ ,  $D_k \sim L_{i_k} - L_{I_k}$ ,  $D_l \sim L_{i_l} - L_{I_l}$ . Then  $I_j \cap I_k \cap I_l = \emptyset$ .*
- 2) *If  $D_j$ ,  $D_k$  are two distinct nonsingular rational curves, then  $\text{Card}(I_j \cap I_k) \leq 1$ .*

Proof: The surface  $S$  can be deformed into a blown-up Hopf surface  $\pi : \mathcal{S} \rightarrow \Delta$  with a flat family  $\mathcal{C}$ , where  $C_0 = C = D_0 + \dots + D_{s-1} \sim -(L_r + \dots + L_{n-1})$ , and  $C_u$ ,  $u \neq 0$ , is an elliptic curve blown-up  $n - r$  times. We denote by  $\Pi = \Pi_{0,u} \dots \Pi_{n-1,u} : S_u \rightarrow S'_u$  the composition of blowing-ups and  $(\Pi_{i+1} \dots \Pi_{n-1})^*(E_{i,u}) \sim L_i$ . We have  $D_j \sim L_{i_j} - L_{I_j}$ ,  $j = 0, \dots, s-1$ , hence  $D_j$  is homologous to an exceptional rational curve of the first kind  $E_{i_j}$  blown-up  $\text{Card}(I_j)$  times. Since an exceptional curve cannot blow-up three rational curves we have the first assertion. Moreover two distinct exceptional rational curve of the kind cannot be blown-up two times by the same curves, whence the second assertion.  $\square$

**Proposition 2. 35** *Let  $S$  be a minimal surface containing a cycle  $C = D_0 + \dots + D_{s-1} \sim -(L_r + \dots + L_{n-1})$  of  $s \geq 1$  nonsingular rational curves. Then,  $r = s$  and numbering properly the line bundles  $L_i$ , for  $i = 0, \dots, r-1$ , we have  $D_i \sim L_i - L_{I_i}$ . Moreover  $\cup_{i=0}^{s-1} I_i = [0, n-1]$  and*

$$\sharp(C) - C^2 = b_2(S).$$

Proof: 1) The case  $s = 1$  has been proved in [21] (see lemma (30)).

2) **If  $s = 2$** ,  $D_0 \sim L_{i_0} - L_{I_0}$ ,  $D_1 \sim L_{i_1} - L_{I_1}$ . We have

$$2 = D_0 D_1 = -L_{i_0} L_{I_1} - L_{i_1} L_{I_0} + L_{I_0} L_{I_1},$$

whence  $i_1 \in I_0$ ,  $i_0 \in I_1$ ,  $I_0 \cap I_1 = \emptyset$ . Setting  $I'_0 = I_0 \setminus \{i_1\}$  and  $I'_1 = I_1 \setminus \{i_0\}$ , we obtain

$$D_0 \sim L_{i_0} - L_{i_1} - L_{I'_0}, \quad D_1 \sim L_{i_1} - L_{i_0} - L_{I'_1}, \quad \text{with } I'_0 \cap I'_1 = \emptyset, \quad \{i_0, i_1\} \cap (I'_0 \cup I'_1) = \emptyset.$$

Therefore

$$-(L_r + \dots + L_{n-1}) \sim C = D_0 + D_1 \sim -(L_{I'_0} + L_{I'_1})$$

i.e.  $I'_0 \cup I'_1 = [r, n-1]$ . Let  $I = \{i_0, i_1\} \cup I'_0 \cup I'_1$  and  $I' = [0, n-1] \setminus I$ . Of course,  $L_{I'} \cdot D_0 = L_{I'} \cdot D_1 = 0$ , whence by (19) 3),  $I' = [0, r-1]$  which is impossible. Therefore  $I' = \emptyset$  and  $\{i_0, i_1\} \cup I'_0 \cup I'_1 = [0, n-1]$ , i.e.  $r = 2$  and

$$\sharp(C) - C^2 = 2 + (n-2) = b_2(S).$$

3) **If  $s \geq 3$** ,  $D_j \sim L_{i_j} - L_{I_j}$ ,  $j = 0, \dots, s-1$  and we may suppose that

$$D_0 D_1 = \dots = D_{s-2} D_{s-1} = D_{s-1} D_0 = 1.$$

Since there is a deformation of  $S$  in which  $D_j + D_{j+1} \sim L_{i_j} + L_{i_{j+1}} - L_{I_j} - L_{I_{j+1}}$  is deformed into a nonsingular rational curve  $D'_u$  in a (perhaps non minimal) surface, contained in a cycle  $C'_u$ , hence  $D_j + D_{j+1}$  must be of the homological type **a**, and either  $i_j \in I_{j+1}$ ,  $i_{j+1} \notin I_j$  or  $i_j \notin I_{j+1}$ ,  $i_{j+1} \in I_j$ . Moreover the equality

$$1 = D_j D_{j+1} = -L_{i_j} L_{I_{j+1}} - L_{i_{j+1}} L_{I_j} + L_{I_j} L_{I_{j+1}}$$

implies that  $I_j \cap I_{j+1} = \emptyset$ , whence

$$\forall j, \quad 1 = D_j D_{j+1} = -L_{i_j} L_{I_{j+1}} - L_{i_{j+1}} L_{I_j}.$$

4) Now, we show by induction on  $s \geq 2$  that for all  $j$ ,  $0 \leq j \leq s-1$ , there is a unique index, denoted  $\sigma(j)$ , such that  $i_j \in I_{\sigma(j)}$ . This assertion is evident if  $s = 2$ , therefore we suppose that  $s \geq 3$ . By 3) it is possible to choose the numbering in such a way that  $i_{s-1} \in I_{s-2}$ . We choose a deformation  $\Pi : \mathcal{S} \rightarrow \Delta$  of  $S$  over the disc endowed with a flat family  $\mathcal{C}$  of curves such that  $C_0 = C$ , and for  $u \neq 0$ ,

$$D'_u \sim D_{s-2} + D_{s-1} \sim L_{i_{s-2}} + L_{i_{s-1}} - (L_{I_{s-2}} + L_{I_{s-1}}) = L_{i_{s-2}} - (L_{I'_{s-2}} + L_{I_{s-1}}),$$



with  $I'_{s-2} = I_{s-2} \setminus \{i_{s-1}\}$ . Consider the cycle  $C_u = D_{0,u} + \cdots + D_{s-3,u} + D'_u$  of  $s-1$  rational curves in the (perhaps non minimal) surface  $S_u$ ; by the induction hypothesis, since  $i_{s-1} \in I_{s-2}$  and  $I'_{s-2} \cap I_{s-1} = \emptyset$ , for all  $0 \leq j \leq s-2$ , there is a unique index  $\sigma(j)$ ,  $0 \leq \sigma(j) \leq s-1$  such that  $i_j \in I_{\sigma(j)}$ . Repeting this argument with  $D_{s-3} + D_{s-2}$  we obtain the result. Therefore we have a well-defined mapping  $\sigma : \{0, \dots, s-1\} \rightarrow \{0, \dots, s-1\}$  such that for all  $j$ ,  $0 \leq j \leq s-1$ ,  $i_j \in I_{\sigma(j)}$ .

5) Setting  $I'_j = I_j \setminus \sigma^{-1}(j)$  we have

$$\begin{aligned} -(L_r + \cdots + L_{n-1}) &\sim C = D_0 + \cdots + D_{s-1} \sim L_{i_0} + \cdots + L_{i_{s-1}} - (L_{I_0} + \cdots + L_{I_{s-1}}) \\ &\sim -(L_{I'_0} + \cdots + L_{I'_{s-1}}) \end{aligned}$$

therefore

$$[r, n-1] = I'_0 \cup \cdots \cup I'_{s-1} \quad \text{with} \quad \{i_0, \dots, i_{s-1}\} \cap (I'_0 \cup \cdots \cup I'_{s-1}) = \emptyset$$

If  $I = \{i_0, \dots, i_{s-1}\} \cup I'_0 \cup \cdots \cup I'_{s-1}$  and  $I' = [0, n-1] \setminus I$ , we have  $L_{I'} D_j = 0$  for all  $j = 0, \dots, s-1$ , then if  $I' \neq \emptyset$ , lemma (19) would imply that  $I' = [0, r-1]$  and this is impossible, hence  $I' = \emptyset$  and  $\{i_0, \dots, i_{s-1}\} \cup I'_0 \cup \cdots \cup I'_{s-1} = [0, n-1]$ , i.e.  $r = s$  and  $\{i_0, \dots, i_{r-1}\} = [0, r-1]$ . Finally

$$\sharp(C) - C^2 = s + (n-r) = n = b_2(S).$$

□

**Corollary 2. 36** *Let  $S$  be a minimal surface with  $b_2(S) \geq 1$ . If  $E$  is a  $k$ -cycle,  $k = \sharp(E) \geq 0$ , such that  $i_\star H_1(E, \mathbb{Z}) = H_1(S, \mathbb{Z})$ , then  $\sharp(E) - E^2 = b_2(S)$ .*

**Remark 2. 37** If  $[H_1(S, \mathbb{Z}) : i_\star H_1(E, \mathbb{Z})] = 2$ ,  $S$  is an odd Inoue-Hirzebruch surface with only one cycle  $C$ . Then we have  $\sharp(C) - C^2 = 2b_2(S)$  (see [20] (2.13)).

For surfaces containing a divisor  $D$  such that  $D^2 = 0$ , i.e. Enoki surfaces, the situation is well understood: They all contain GSS,  $D$  is a cycle of  $b_2(S)$  rational curves  $D = D_0 + \cdots + D_{n-1}$  such that  $D_0^2 = \cdots = D_{n-1}^2 = -2$  and  $S$  admits a numerically anticanonical divisor if and only if  $S$  is a Inoue surface, in which case  $h^0(S, -K) = 1$ . For surfaces such that  $D^2 < 0$  for any divisor we have the following theorem:

**Theorem 2. 38** *Let  $S$  be a compact complex surface of class  $VII_0^+$ . We suppose that there is no divisor such that  $D^2 = 0$ . Then the following properties are equivalent:*

- i)  $S$  contains a GSS,
- ii)  $S$  admits a NAC divisor.
- iii)  $S$  contains  $b_2(S)$  rational curves,

Proof:  $i) \Leftrightarrow iii)$  by [12].

$i) \Rightarrow ii)$  The intersection matrix  $M(S)$  is negative definite and the rational curves give a  $\mathbb{Q}$ -base of  $H^2(S, \mathbb{Q})$ , whence there is an integer  $m \geq 1$  and a divisor  $D_m$  such that in  $H^2(S, \mathbb{Z})$ ,  $mK + D_m = 0$ . By lemma (22),  $D_m$  is effective.

$ii) \Rightarrow i)$  By lemma (24) we may suppose that the index satisfies  $m(S) = 1$ , i.e.  $S$  has a NAC divisor  $D_{-K}$ . By theorem (30) we may suppose that there is no singular curve. By lemma (27) there are two cases:  $S$  is a Inoue-Hirzebruch surface, in particular contains a GSS, or  $S$  contains a cycle with at least one branch. Therefore we have to prove the result in the second case. Let  $D = \sum_{i=0}^p D_i$  be the maximal divisor, where  $C = D_0 + \dots + D_{s-1}$  is the cycle. By (33), we have in  $H^2(S, \mathbb{Z})$ ,

$$-(L_0 + \dots + L_{n-1}) = -K = \sum_{i=0}^p k_i D_i = \sum_{i=0}^p k_i (L_i - L_{I_i}),$$

where  $k_i \geq 1$ . If  $p < n - 1$ , the curve  $D_{n-1}$  is missing, hence there is exactly one index  $j$  such that  $n - 1 \in I_j$  and  $k_j = 1$ . Moreover by Proposition (35),  $\cup_{i=0}^{s-1} I_i = [0, n - 1]$ , therefore  $j \leq s - 1$ . By lemma (26),  $S$  would be a Inoue-Hirzebruch surface, which is a contradiction.  $\square$

**Corollary 2. 39 ([19])** *Let  $S$  be a surface of class  $VII_0$  with  $b_2(S) > 0$ . Then  $S$  is a Inoue-Hirzebruch surface if and only if there exists two twisted vector fields  $\theta_1 \in H^0(S, \Theta \otimes F_1)$ ,  $\theta_2 \in H^0(S, \Theta \otimes F_2)$ , where  $F_1, F_2$  are flat line bundles, such that  $\theta_1 \wedge \theta_2(p) \neq 0$  at at least one point  $p \in S$ .*

Proof:  $\theta_1 \wedge \theta_2$  is a non trivial section of  $-K \otimes F$  whence  $S$  contains a GSS. We conclude by [10] th. 5.5.  $\square$

### 3 On classification of bihermitian surfaces

#### 3.1 Conformal and complex structures

We consider connected oriented conformal 4-manifolds  $(M^4, c)$  with two complex (i.e. integrable almost-complex) structures  $J_1, J_2$  which induce the same orientation. Given a riemannian metric  $g$  in  $c$ ,  $(M^4, g, J_1, J_2)$  is called a **bihermitian surface** relatively to the conformal class  $c$  if

- $J_i$  are orthogonal with respect to the metric, i.e.  $g(J_i X, J_i Y) = g(X, Y)$ ,  $i = 1, 2$ ,
- $J_1$  and  $J_2$  are independent, i.e. there is a point  $x \in M$  such that  $J_1(x) \neq \pm J_2(x)$ .

The triple  $(c, J_1, J_2)$  is called a bihermitian structure on  $M^4$ . If moreover  $J_1(x) \neq \pm J_2(x)$  everywhere, the bihermitian structure  $(c, J_1, J_2)$  is called **strongly bihermitian**.

Given two such almost-complex structures  $J_1$  and  $J_2$ , denote by  $f$  the smooth function, called the angle function

$$f = \frac{1}{4} \text{tr}({}^t J_1 J_2) = -\frac{1}{4} \text{tr}(J_1 J_2).$$

By Cauchy-Schwarz inequality,  $|f| \leq 1$  and  $f(x) = \pm 1$  if and only if  $J_1(x) = \pm J_2(x)$ . Since  $J_1 J_2 \in SO(4)$  and  $J_2 J_1$  is the inverse of  $J_1 J_2$ , it is easy to check that

$$J_1 J_2 + J_2 J_1 = -2f \text{Id}.$$

Moreover  $J_1(x)$  and  $J_2(x)$  anticommute if and only if  $f(x) = 0$ .

Another conformal structure is provided by the Weyl curvature tensor: The riemannian curvature tensor  $R$  of type (3,1) has a classical decomposition, under the orthogonal group  $O(4)$ , into three parts given by the scalar curvature, the Ricci curvature tensor without trace and the Weyl curvature tensor  $W$  of type (3,1) which is a conformal invariant [4].

Let  $*$  :  $\bigwedge^2 T^*M \rightarrow \bigwedge^2 T^*M$  be the star-Hodge operator, with  $\star^2 = \text{Id}$  and two eigenvalues  $\pm 1$ . We denote by  $\bigwedge_+^2$  (resp.  $\bigwedge_-^2$ ) the eigenspace associated to  $+1$  (resp.  $-1$ ). Since  $*$  depends only on  $c$ , the splitting  $\bigwedge^2 T^*M = \bigwedge_+^2 \oplus \bigwedge_-^2$  is a conformal invariant. Let  $W(g) : \bigwedge^2 T^*M \rightarrow \bigwedge^2 T^*M$  be the Weyl curvature tensor of type (2,2), with restrictions  $W_\pm \in \text{End}(\bigwedge_\pm^2)$  over  $\bigwedge_\pm^2$ . The riemannian conformal class of  $(M^4, g)$  is called **anti-self-dual (ASD)** if  $*W(g) = -W(g)$ , or equivalently  $W_+ = 0$ .

The geometric meaning of ASD condition stems from Atiyah-Hitchin-Singer theorem [3]: Let

$$Z = \{J \in SO(TM) \mid J^2 = -\text{Id}\} = SO(TM)/U(2) \rightarrow M$$

be the twistor space, i.e. the space of all orthogonal almost-complex structures over  $(M, g)$ , inducing the orientation of  $M$ . The fiber is isomorphic to the Riemann sphere  $S^2$ . Since there is only one complex structure on  $S^2$ ,  $Z$  is a differentiable fiber bundle with complex fiber  $\mathbb{P}^1(\mathbb{C})$ . The complex structure on the fiber and on the base yield a canonical almost-complex structure  $\mathbb{J}$  of the twistor space  $Z$ , which is not integrable in general. The theorem of Atiyah-Hitchin-Singer asserts that  $\mathbb{J}$  is integrable if and only if the metric  $g$  is ASD, hence any compatible complex structure at a point  $x \in M$  extends into a compatible complex structure over a neighbourhood of  $x$ . However, it does not extend to the whole manifold and perhaps  $(M, c)$  admits no complex structure (for instance  $S^4$ ). **The aim is to classify compact 4-manifolds with several compatible complex structures, or at least to give necessary conditions for their existence.** When there are more than two compatible almost structures, we have

**Proposition 3. 40 ([22])** *If an oriented riemannian 4-manifold  $(M, g)$  admits three independent compatible complex structures then the metric  $g$  is anti-self-dual.*

By Pontecorvo's classification of ASD bihermitian surfaces [22] Prop. 3.7, these surfaces are hyperhermitian. Following C.P. Boyer [4] we define **hyperhermitian complex surfaces** as oriented compact conformal 4-manifolds  $(M, c, \mathcal{F})$  with a 2-sphere  $\mathcal{F}$  of compatible complex structures generated by two anti-commuting ones. A hyperhermitian 4-manifold  $(M, c, \mathcal{F})$  must be one of the following

- A flat complex torus
- A K3 surface with Ricci-flat Kähler metric, or
- A special Hopf surface

in particular they all have  $b_2(M) = 0$ . We refer to [22] and [4] for details. Now we focus on the case where first Betti number  $b_1(M)$  is odd; for the even case see [2].

### 3.2 Numerically anticanonical divisor of a bihermitian surface

For the convenience of the reader we recall the results used in the sequel (see [1, 2, 22]): Denote by

$$F_i^g(\cdot, \cdot) = g(J_i \cdot, \cdot), \quad i = 1, 2$$

the Kähler forms of  $(g, J_1)$  and  $(g, J_2)$  respectively,  $\theta_1^g, \theta_2^g$  their Lee forms, i.e.

$$dF_i^g = \theta_i^g \wedge F_i^g, \quad i = 1, 2$$

We furthermore denote by  $[J_1, J_2] = J_1 J_2 - J_2 J_1$  the commutator of  $J_1$  and  $J_2$  and we consider the real  $J_i$ -anti-invariant 2-form

$$\Phi^g(\cdot, \cdot) = \frac{1}{2}g([J_1, J_2] \cdot, \cdot),$$

and the corresponding complex  $(0, 2)$ -forms

$$\sigma_i^g(\cdot, \cdot) = \Phi^g(\cdot, \cdot) + i\Phi^g(J_i \cdot, \cdot).$$

Then  $\sigma_i^g$ ,  $i = 1, 2$  are smooth sections of the anti-canonical bundle  $K_{J_i}^{-1} \simeq \bigwedge_{J_i}^{(0,2)}(M)$  of  $(M, J_i)$  and  $\sigma_i^g(x) = 0$  if and only if  $\Phi^g(x) = 0$  if and only if  $J_1(x) = \pm J_2(x)$ . Therefore the common zero set of  $\sigma_i$  is exactly  $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$ , where

$$\begin{aligned} \mathcal{D}^+ &= f^{-1}(1) = \{x \in M \mid J_1(x) = J_2(x)\}, \\ \mathcal{D}^- &= f^{-1}(-1) = \{x \in M \mid J_1(x) = -J_2(x)\}. \end{aligned}$$

**Lemma 3. 41** ([1, 2]) *Let  $(M, c, J_1, J_2)$  be a bihermitian surface. Then, for any metric  $g$  in the conformal class  $c$ , the 1-forms  $\theta_1^g, \theta_2^g$  and  $\sigma_1^g$  satisfy the following properties:*

i) *If  $M$  is compact, then  $d(\theta_1^g + \theta_2^g) = 0$ ,*

ii)  *$\bar{\partial}_{J_1} \sigma_1^g = -\frac{1}{2}(\theta_1^g + \theta_2^g)^{(0,1)} \otimes \sigma_1^g$ ,*

*where  $(\cdot)^{(0,1)}$  denotes the  $(0, 1)$  part and  $\bar{\partial}_{J_1}$  is the Cauchy-Riemann operator relatively to  $J_1$ .*

Proof: i) By [2],  $(d(\theta_1^g + \theta_2^g))_+ = 0$ . Let  $\delta = -\star d\star$  be the adjoint of  $d$ . Setting  $\varphi = \theta_1^g + \theta_2^g$ ,  $\Delta d\varphi = d\delta d\varphi = -d\star d\star d\varphi = 0$ , for by assumption  $\star d\varphi = -d\varphi$ . Hence  $\delta d\varphi = 0$ , and  $|d\varphi|^2 = (d\varphi, d\varphi) = (\varphi, \delta d\varphi) = 0$ .

ii) The proof in [2] Lemma 3 is local.  $\square$

**Proposition 3. 42** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface. Then there exists a topologically trivial line bundle  $L \in H^1(M, \mathbb{R}_+^\star)$  such that  $\sigma_i$  is a non-trivial holomorphic section of  $K_{J_i}^{-1} \otimes L$ , in particular*

$$H^0(M, K_{J_i}^{-1} \otimes L) \neq 0, \quad i = 1, 2,$$

and  $\mathcal{D}^+, \mathcal{D}^-$  are empty or complex curves for both  $(M, J_1)$  and  $(M, J_2)$ .

Moreover, if  $(c, J_1, J_2)$  is strongly bihermitian, then  $K_{J_i} = L$ , in particular,  $b_2(M) = 0$ .

Proof: By (41),  $\theta_1^g + \theta_2^g$  is closed, hence there is an open covering  $(U_j)_{j \in I}$  and  $\mathcal{C}^\infty$  functions  $\Phi_j : U_j \rightarrow \mathbb{R}$ , such that for the local metric  $g_j = \exp(\Phi_j)g$ , the local Lee forms satisfy  $\theta_1^{g_j} + \theta_2^{g_j} = 0$ , i.e.  $(\theta_1^g + \theta_2^g)|_{U_j} = -2d\Phi_j$ . Setting  $c_{jk} = \exp(\Phi_j - \Phi_k) \in \mathbb{R}_+^\star$ ,  $\sigma_i^j = \sigma_i^{g_j}$ , we obtain a topologically trivial line bundle  $L = [(c_{jk})] \in H^1(M, \mathbb{R}_+^\star)$ , and a holomorphic section  $\sigma_i = (\sigma_i^j)$  of  $K_{J_i}^{-1} \otimes L$ . If  $(c, J_1, J_2)$  is strongly bihermitian  $\mathcal{D} = \emptyset$ , therefore  $\sigma_i$  is a non-vanishing holomorphic section of  $K_{J_i}^{-1} \otimes L$ .  $\square$

### 3.3 Bihermitian surfaces with odd first Betti number

The Kodaira dimension of a compact bihermitian surface with odd first Betti number is  $\kappa = -\infty$  ([1] thm 1), hence the minimal models of  $(M, J_1)$  and of  $(M, J_2)$  are in class VII<sub>0</sub>. In this section, we shall complete (and simplify) the classification theorem of V. Apostolov [1]. We need first

**Lemma 3. 43** *Let  $S$  be a complex surface and let  $\Pi : S' \rightarrow S$  be the blowing-up of  $x \in S$ , and  $E = \Pi^{-1}(x)$  its exceptional curve.*

1) *If there exists a flat line bundle  $L'$  on  $S'$  such that  $H^0(S, K_S^{-1} \otimes L') \neq 0$ , then for  $L = \Pi_\star L'$ ,  $H^0(S, K_S^{-1} \otimes L) \neq 0$ .*

2) *If there exists a flat line bundle  $L$  on  $S$  such that  $H^0(S, K_S^{-1} \otimes L) \neq 0$ , and if  $\Pi$  blows-up a point on the effective twisted anticanonical divisor, then for  $L' = \Pi^\star L$ ,  $H^0(S', K_{S'}^{-1} \otimes L') \neq 0$ .*

Proof: 1) The coherent sheaf  $L = \Pi_\star L'$  is locally trivial since on a simply connected neighbourhood  $U$  of the exceptional curve  $E$ ,  $L|_U$  is trivial. The line bundle  $K_S^{-1} \otimes L$  has a section on  $S \setminus \Pi(E)$  which extends by Hartogs theorem. 2) We have  $K_{S'}^{-1} \otimes L' = \pi^\star(K_S^{-1} \otimes L) - E$  hence a section  $\sigma$  of  $K_S^{-1} \otimes L$  yields a section of  $K_{S'}^{-1} \otimes L'$  if and only if  $x$  belongs to the zero set of  $\sigma$ .  $\square$

**Theorem 3. 44** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface with odd first Betti number.*

1) If  $(M, c, J_1, J_2)$  is strongly bihermitian (i.e  $\mathcal{D} = \emptyset$ ), then the complex surfaces  $(M, J_i)$  are minimal and either a Hopf surface covered by a primary one associated to a contraction  $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  of the form

$$F(z_1, z_2) = (\alpha z_1 + s z_2^m, a \alpha^{-1} z_2),$$

$$\text{with } a, s \in \mathbb{C}, 0 < |\alpha|^2 \leq a < |\alpha| < 1, (a^m - \alpha^{m+1})s = 0,$$

or else  $(M, J_i)$  are Inoue surfaces  $S_{N,p,q,r;t}^+$ ,  $S_{N,p,q,r}^-$ .

2) If  $(M, c, J_1, J_2)$  is not strongly bihermitian, then  $\mathcal{D}$  has at most two connected components,  $(M, J_i)$ ,  $i = 1, 2$ , contain GSS and the minimal models  $S_i$  of  $(M, J_i)$  are

- Surfaces with GSS of intermediate type if  $\mathcal{D}$  has one connected component
- Hopf surfaces of special type (see [22] 2.2), Inoue (parabolic) surfaces or Inoue-Hirzebruch surfaces if  $\mathcal{D}$  has two connected components.

Moreover, the blown-up points belong to the NAC divisors.

Proof: 0) Since the fundamental group of a surface with GSS is isomorphic to  $\mathbb{Z}$ , since  $\pi_1(S_{N,p,q,r;t}^{(+)}) \neq \mathbb{Z}$ ,  $\pi_1(S_{N,p,q,r}^{(-)}) \neq \mathbb{Z}$ , and  $S_{N,p,q,r}^{(-)}$  is a quotient of a surface  $S_{N,p,q,r;t}^{(+)}$  ([15] p 276),  $(M, J_1)$  and  $(M, J_2)$  must be of the same type.

1) If  $(M, c, J_1, J_2)$  is strongly bihermitian,  $(M, J_i)$  are minimal using (42). Applying the classification of minimal surfaces of class  $\text{VII}_0$  with  $b_2(S) = 0$  [18], we have to consider Hopf surfaces and Inoue surfaces. We derive from Proposition (42) that  $-K_{J_i} \in H^1(M, \mathbb{R}_+^*)$ . On one hand the anticanonical line bundle of a surface  $S_M$  is not real (11), hence we may exclude it. On second hand, a finite covering of a bihermitian surface is bihermitian and if a primary Hopf surface is associated to the contraction

$$F(z) = (\alpha_1 z_1 + s z_2^m, \alpha_2 z_2), \quad 0 < |\alpha_1| \leq |\alpha_2| < 1, \quad (\alpha_2^m - \alpha_1)s = 0,$$

$-K = L^{\alpha_1 \alpha_2}$ , therefore  $a = \alpha_1 \alpha_2 \in \mathbb{R}_+^*$  and it is easy to check that the requested conditions are fulfilled.

2) If  $(M, c, J_1, J_2)$  is not strongly bihermitian,  $(M, J_i)$  admits a non-trivial effective NAC divisor  $D_{-K_i}$  whose support is  $\mathcal{D}$ , and as noticed in [1], 3.3, it is the same for the minimal model  $S_i$  of  $(M, J_i)$ . Using theorem (38),  $S_i$  contains GSS and the support of the NAC divisor has at most two components by (29). If  $\Pi = \Pi_m \cdots \Pi_0 : (M, J_i) \rightarrow S_i$  is the blowing-down of the exceptional curves, the blown-up points belong to the successive NAC divisors by (43), hence  $\mathcal{D}$  has at most two connected components.  $\square$

**Corollary 3. 45** *Let  $(M, c, J_1, J_2)$  be a compact ASD bihermitian surface with odd first Betti number. Then the minimal models of the complex surfaces  $(M, J_i)$ ,  $i = 1, 2$ , are*

- Hopf surfaces of special type (see [22] 2.2),
- (parabolic) Inoue surfaces or

- even Inoue-Hirzebruch surfaces.

Moreover, the blown-up points belong to the NAC divisors.

Proof: By [22] 3.11,  $(M, J_i)$  has a minimal model  $S_i$  in the class  $\text{VII}_0$  and

$$H^0(M, -K_{J_i}) \neq 0,$$

therefore  $S_i$  is not a Inoue surface  $S_{N,p,q,r;t}^{(+)}$ ,  $S_{N,p,q,r}^{(-)}$ . If  $b_2(S_i) = 0$ ,  $S_i$  is a Hopf surface, and if  $b_2(S_i) > 0$ ,  $S_i$  contains a GSS by (38). The existence of a (non-twisted) global section of  $-K$  is equivalent to the existence of a metric  $g \in c$  such that  $\theta_1^g + \theta_2^g = 0$ . In this situation, [2] Prop.4 asserts that  $\mathcal{D}_+$  and  $\mathcal{D}_-$  are both non-empty. This is possible only when  $S_i$  is parabolic Inoue or an even Inoue-Hirzebruch surface.  $\square$

## References

- [1] V. APOSTOLOV: Bihermitian surfaces with odd first Betti number. *Math. Z.* 238 (2001), 555-568
- [2] V. APOSTOLOV, P. GAUDUCHON, G. GRANTCHAROV : Bihermitian structures on complex surfaces. *Proc. London Math. Soc.* 79, 414-428 (1999)
- [3] M.F. ATIYAH, N.J. HITCHIN, & I.M. SINGER : Self-duality in 4-dimensional Riemannian Geometry. *Proc. Roy. Soc. London, Ser. A* 362 (1978) 425-461.
- [4] C.P. BOYER : A note on hyperhermitian manifolds. *Proc. Amer. Math. Soc.* 102, 157-164 (1988)
- [5] W. BARTH, C. PETERS, A. VAN DE VEN : Compact Complex Surfaces. *Springer* (1984).
- [6] G. DLOUSSKY : Structure des surfaces de Kato. *Mémoire de la S.M.F* 112.n°14 (1984).
- [7] G. DLOUSSKY : Une construction élémentaire des surfaces d’Inoue-Hirzebruch. *Math. Ann.* 280, (1988), 663-682.
- [8] G. DLOUSSKY : Complex surfaces with Betti Numbers  $b_1 = 1$ ,  $b_2 > 0$  and finite quotients. *Contemp. Math. (AMS)* 288, (2001), 305-309.
- [9] G. DLOUSSKY & F. KOHLER : Classification of singular germs of mappings and deformations of compact surfaces of class  $\text{VII}_0$ . *Ann. Polonici Math. LXX* (1998), 49-83
- [10] G. DLOUSSKY, K. OELJEKLAUS : Vector fields and foliations on compact surfaces of class  $\text{VII}_0$ . *Ann. Inst. Fourier* 49, 5 (1999), 1503-1545

- [11] G. DLOUSSKY, K. OELJEKLAUS, M. TOMA : Surfaces de la classe  $VII_0$  admettant un champ de vecteurs. *Comment. Math. Helvet.* 76 (2001), 640-664
- [12] G. DLOUSSKY, K. OELJEKLAUS, M. TOMA : Class  $VII_0$  surfaces with  $b_2$  curves. *Tohoku Math. J.* 55 (2003), 283-309
- [13] S.K. DONALDSON : The orientation of Yang-Mills moduli space and 4-manifolds topology. *J. Differential Geometry* 26 (1987) 397-428.
- [14] I. ENOKI : Surfaces of class  $VII_0$  with curves. *Tôhoku Math. J.* 33, (1981), 453-492.
- [15] M. INOUE On surfaces of Class  $VII_0$ , *Inventiones math.* 24 (1974) 269-310.
- [16] K. KODAIRA On stability of compact submanifolds of complex manifolds. *Ann.Math.* 85 (1963), 79-94.
- [17] K. KODAIRA On the structure of compact complex analytic surfaces I, II. *Am. J. of Math.* vol.86, 751-798 (1964); vol.88, 682-721 (1966).
- [18] M. LÜBKE , A. TELEMAN: The Kobayashi-Hitchin Correspondance. *World Scientific* (1995).
- [19] K. OELJEKLAUS, M. TOMA & D. ZAFRAN: Une caractérisation des surfaces d'Inoue-Hirzebruch. *Ann. Inst. Fourier* 51, 5 (2001), 1243-1257.
- [20] I. NAKAMURA : On surfaces of class  $VII_0$  with curves. *Invent. Math.* 78,(1984), 393-443.
- [21] I. NAKAMURA On surfaces of class  $VII_0$  with curves II. *Tohoku Math. J.* 42 (1990), 475-516.
- [22] M. PONTECORVO : Complex structures on Riemannian four-manifolds. *Math. Ann.* 309, 159-177 (1997)

Centre de Mathématiques et d'Informatique  
 Laboratoire d'Analyse Topologie et Probabilités  
 Université d'Aix-Marseille 1  
 39, rue F. Joliot-Curie  
 13453 Marseille Cedex 13  
 FRANCE